Interpolating Sequences and Extremal Problems

by

Ching-Cheong Lee

A Thesis Submitted to
The Hong Kong University of Science and Technology
in Partial Fulfillment of the Requirements for
the Degree of Master of Philosophy
in Mathematics

15 August 2014, Hong Kong

Authorization

I hereby declare that I am the sole author of the thesis.

I authorize the University of Science and Technology to lend this thesis to other institutions or individuals for the purpose of scholarly research.

I further authorize the University of Science and Technology to reproduce the thesis by photocopying or by other means, in total or in part, at the request of other institutions or individuals for the purpose of scholarly research.

Cheong

Ching-Cheong Lee

Interpolating Sequences and Extremal Problems

by

Ching-Cheong Lee

This is to certify that I have examined the above MPhil thesis
and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by
the thesis examination committee have been made.

Prof. Kin-Yin Li, Thesis Supervisor

Prof. Yang WANG, Head of Department

Department of Mathematics

15 August 2014

Acknowledgment

My first sincere gratitude must go to my supervisor, Professor Kin-Yin Li, who has given me a lot of advice and help in pursuing my postgraduate study. He has also brought me into a particular field of research and provided helpful guidance in my research during the two years of my MPhil study.

I am thankful to the Department of Mathematics for providing me with the opportunity to study in and outside HKUST. Also I would like to thank the PG students in the Department of Mathematics who have given me a lot of fun, help and care. Special thanks go to my TA fellows Ho-Yin Ho, Yu-Sing Hon and Enoch Kung, who have been my comrade in various important public exams.

Lastly, but definitely not least, I wish to say a big thank-you to almost every Instructional Assistant in the Department of Mathematics, particularly Cheuk-Yin Au, Yun-Hung Chan, Hiu-Hung Huang, Hing-Sang Lau, Ho-Ming Leung, Xing Li, Xilin Li, Chi-Yan So and finally Kwok-Pang Wong, for providing the PG student room a joyful working atmosphere during the past two years.

Contents

Interpolating Sequences and Extremal Problems

Ching-Cheong Lee

Department of Mathematics

Abstract

In 1958 Carleson proved that a sequence is interpolating if it is uniformly separated, later in 1983 Peter Jones gave a simple constructive proof in [JON] which is stunning in his time. On the other hand, when a sequence is not uniformly separated, Garnett gave a sharp positive result in [GAR2]. Following Jones's idea in [JON] and also a highly relevant paper [VIN] due to Vinogradov, we give a constructive proof to Garnett's result on nonuniformly separated sequence with slightly modified assumption. We will also discuss extremal problems that is described in the language of Hardy spaces and solve it in functional analytical point of view.

Chapter 1

Preliminaries

In Section 1.1 and 1.2 we will mention all necessary definitions and background in complex analysis that we are going to use throughout the thesis. In Section 1.3 we give a complete introduction to interpolating sequences and known results in this aspect, including the most important result—the Carleson Theorem—that we use and partially generalize in the next chapter.

1.1 Notations

In this thesis we denote

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} =: D(0,1) \quad \text{and} \quad T = \{|z| = 1\} = [0, 2\pi)$$

the open unit disk and its boundary respectively. We denote $\dfrac{dt}{2\pi}$ the normalized Lebesgue measure on $T = [0, 2\pi)$.

Given an analytic function $f : \mathbb{D} \to \mathbb{C}$ and a fixed $r \in [0, 1)$, we denote

$$f_r(z) = f(rz), \quad z \in \frac{1}{r}\mathbb{D}$$

the r-dilation of $f(z)$. We also use the notation

$$f_r(\theta) := f(re^{i\theta})$$

which is a continuous function on the circle T.

Whenever a function is defined on T, we use $\|\cdot\|_p$ to denote the p-norm on $L^p(T, d\theta/2\pi)$, that is,

$$\|f_r\|_p = \left(\int_0^{2\pi} |f_r(\theta)|^p \frac{d\theta}{2\pi} \right)^{1/p}.$$

The notation $\|\cdot\|_p$ will carry another meaning in (1.3). This is the only extra meaning and hopefully no confusion will be caused.

1.2 Hardy Spaces

1.2.1 As a Subspace of $L^p(T, d\theta/2\pi)$

The prime function space on which we do analysis is:

Definition 1.1. For $p > 0$, the notation $H^p(\mathbb{D})$ (or simply H^p) denotes the class of analytic functions $f(z)$ on $\mathbb{D}$ satisfying

$$\sup_{0 \le r < 1} \|f_r\|_p < \infty. \tag{1.2}$$

The set of function in H^p is called the *Hardy space* on $\mathbb{D}$.

When $p = \infty$, the definition says that H^∞ consists of the class of bounded analytic functions. Suppose $f(z)$ is analytic on $\mathbb{D}$, then $|f(z)|^p$ is *subharmonic*, therefore $r \mapsto \|f_r\|_p$ is an increasing function (for a proof, see pp. 336-337 of [RUD]). Thus in the definition of Hardy class functions the supremum is actually the limit of $\|f_r\|_p$ as $r \to 1$. We define

$$\|f\|_p = \lim_{r \to 1} \|f_r\|_p = \lim_{r \to 1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}. \tag{1.3}$$

When $p \ge 1$, the triangle inequality for $L^p(T)$ functions shows that $\|f\|_p$ in (1.3) defines a norm on H^p. When $0 < p < 1$, $\|f\|_p$ is not a norm on L^p any more. In fact it is easy to show conversely that if $\|\cdot\|_p$ satisfies triangle inequality on $L^p(T)$, then $p \ge 1$. However, for $p \in (0, 1)$, the fact that

$$\|fg\|_1 \ge \|f\|_p \|g\|_{p^*}, \quad p^* = p/(p-1)$$

enables us to show

$$d_p(f,g) := \|f-g\|_p^p$$

defines a metric on L^p.

Theorem 1.4. *When $p \geq 1$, H^p is a Banach space with $\|\cdot\|_p$ defined in (1.3). When $0 < p < 1$, the metric $d(f,g) := \|f-g\|_p^p$ turns H^p into a complete metric space (or even more, a Fréchet space).*

An important feature of Hardy space H^p $(p > 0)$ is that it can be identified as a closed subspace of $L^p(T)$, as shown in the following theorem:

Theorem 1.5 ([RUD], p. 340). *If $0 < p < \infty$ and $f \in H^p$, then*

(a) The nontangential limits $f^*(e^{i\theta})$ exist a.e. on T and $f^* \in L^p(T)$,

(b) $\lim_{r\to 1} \|f^* - f_r\|_p = 0$ and

(c) $\|f^*\|_p = \|f\|_p$.

As a convention, from now on we will denote $f(t)$ the nonnganential limit instead of f^* throughout the thesis.

Theorem 1.5 tells us that given an $f(z) \in H^p$, we get an $f(t) \in L^p(T)$. To recover back from a function on T to a function on $\mathbb{D}$, we need the following:

Definition 1.6. For $z \in \mathbb{D}$, we define the *Poisson kernel* of $\mathbb{D}$ by

$$P_z(t) = \frac{1-|z|^2}{|e^{it}-z|^2} = \mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right) = \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} =: P_r(\theta-t).$$

The *Poisson integral* of $f : T \to \mathbb{C}$ is *the* harmonic function on $\mathbb{D}$ defined by

$$P[f](z) = \int_0^{2\pi} P_z(t) f(t)\, \frac{dt}{2\pi} = P_r * f(\theta), \quad z = re^{i\theta}.$$

Our notations P_z and P_r are consistently used. It is also not hard to see

$$L^1(T) \to \{\text{harmonic function on } \mathbb{D}\}; \quad f \mapsto P[f]$$

is one-one. The reason is simple: suppose $P[f] = 0$, then for every $g \in C(T)$, Fubini's Theorem tells us $\int_0^{2\pi} P[g]_r f\, dt = \int_0^{2\pi} P[f]_r g\, dt = 0$, by taking $r \to 1$, $\int_0^{2\pi} gf\, dt = 0$, thus we conclude $f = 0$ a.e..

The Poisson kernel of $\mathbb{D}$ satisfies for every $z \in \mathbb{D}$,

$$\int_0^{2\pi} P_z(\theta)\, \frac{d\theta}{2\pi} = 1.$$

Generally the Poisson integral of an L^1 function $f(t)$ on T is harmonic but not analytic, e.g., take $f(z) = z^{-1}$, then $g := P[f|_T]$ cannot be an analytic function. To see this, suppose g were analytic, then $g \in H^1$ by Theorem 1.7 below and later by Lemma 3.1, $\int_0^{2\pi} g(t)e^{int}\, dt = 0$ for every $n \geq 1$, a contradiction arises when $n = 1$ since $g(t) = e^{-it}$.

Although analyticity of functions in the image of $P[\cdot]$ cannot be guaranteed, they automatically satisfy (1.2):

Theorem 1.7. *If $1 \leq p \leq \infty$ and $f \in L^p(T, d\theta/2\pi)$, then*

$$\sup_{0 \leq r < 1} \|P[f]_r\|_p \leq \|f\|_p.$$

This follows easily from the generalized Minkowski inequality:

$$\left\| \int F(x,t)\, d\nu(x) \right\|_{L^p(d\mu(t))} \leq \int \|F(x,t)\|_{L^p(d\mu(t))}\, d\nu(x)$$

which is a standard exercise in real analysis as it can be proved by elementary use of Hölder's inequality.

Theorem 1.7 and density of continuous functions in $L^p(T)$ now give:

Theorem 1.8 ([RUD], p.239). *Let $f \in L^p(T)$, $1 \leq p < \infty$, then*

$$\lim_{r \to 1} \|P[f]_r - f\|_p \to 0.$$

We can state for which functions analytic on $\mathbb{D}$ the integral representation on the boundary is possible:

Theorem 1.9. *A function $f(z)$ analytic in $|z| < 1$ is representable in the form*

$$f(z) = \int_0^{2\pi} P_z(t)\varphi(t)\,\frac{dt}{2\pi}$$

for some $\varphi \in L^1$ if and only if $f \in H^1$. In this case, $\varphi(t) = f^*(e^{it})$ a.e..

For a proof, see p. 31 of [DUR].

When $p \geq 1$, Theorem 1.5 shows that every $f \in H^p$ can be identified with an $f^* \in L^p(T)$ isometrically. Therefore $f^* = 0$ a.e. implies $f = 0$. In fact the zeros of f^* on the boundary must be very "thin". More precisely, Theorem 1.10 below implies that if $f^*(e^{i\theta}) = 0$ on a set of positive measure, then $f \equiv 0$.

Theorem 1.10. *If $f \in H^p, p > 0$, then*

$$\ln|f(z)| \leq \int_0^{2\pi} \ln|f(t)| P_z(t)\,\frac{dt}{2\pi}. \tag{1.11}$$

For a proof using Fatou's lemma, see p. 23 of [DUR], p. 62 of [GAR] or p. 344 of [RUD]. For a proof using subharmonicity of $\ln|f(z)|$, see p. 35 of [RAN].

Direct application of Jensen's inequality to Theorem 1.10 gives:

Corollary 1.12. *If $f \in H^p, p > 0$, then*

$$|f(z)|^p \leq \int_0^{2\pi} |f(t)|^p P_z(t)\frac{dt}{2\pi}.$$

This inequality says that every bounded sequence $\{f_n\}$ in H^p must be locally uniformly bounded on $\mathbb{D}$ and thus forms a normal family. Moreover, this inequality says that the pointwise evaluation at z, denoted by i_z, is a bounded linear functional on H^p. Perhaps the most useful consequence resulting from Corollary 1.12 in this regard is:

$$|f(z)| \leq \left(\frac{1+|z|}{1-|z|}\right)^{1/p} \|f\|_p.$$

We will sharpen the constant $\big((1+|z|)/(1-|z|)\big)^{1/p}$ to $(1-|z|^2)^{1/p}$ in Theorem 3.6 through techniques in solving extremal problems.

1.2.2 Canonical Factorization via Inner and Outer Factors

Definition 1.13. An *inner function* is a bounded analytic $f(z)$ on $\mathbb{D}$ such that $|f(e^{i\theta})| = 1$ for a.e. $\theta \in T$. A *singular inner function* is a nonvanishsing inner function.

Definition 1.14. An *outer function* for the class H^p is a function of the form

$$F(z) = e^{i\gamma} \exp\left\{ \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \ln \varphi(t) \frac{dt}{2\pi} \right\},$$

where γ is a real number, $\varphi \geq 0$, $\ln \varphi \in L^1$ and $\varphi \in L^p$.

We note that inner functions and outer functions are respectively closed under multiplication.

Example 1.15 (Inner Functions). Consider

$$f(z) = \exp\left(\frac{z+1}{z-1}\right),$$

which is an analytic function on $\mathbb{D}$, for every $\zeta = e^{i\theta} \neq 1$ we have

$$|f(\zeta)| = \exp\left\{ \mathrm{Re}\left(\frac{\zeta+1}{\zeta-1}\right)\right\} = \exp\left\{ \mathrm{Re}\left(\frac{-2i\sin\theta}{|1-e^{i\theta}|^2}\right)\right\} = 1,$$

therefore $|f(\zeta)| = 1$ for a.e. $\zeta \in T$, and hence f is an inner function.

Since $f(e^{-i\theta_0} z)$ is again an inner function, for every $\zeta \in T$, the function

$$\exp\left(\frac{z+\zeta}{z-\zeta}\right),$$

is inner. It is evident from the definition that product of two inner functions is again inner, so for $\zeta_1, \ldots, \zeta_n \in T$ the function

$$\exp\left(\sum_{i=1}^{n} \frac{z+\zeta_i}{z-\zeta_i}\right)$$

is also an inner function. ∎

Just like outer functions, inner functions will have a general from described in Theorem 1.17 in terms of singular measures (these are measures concentrated on a set of Lebesgue measure zero), in fact we have just chosen $\mu = \delta_1$ in Example 1.15.

Example 1.16 (Outer Functions). Explicit examples of outer function can be obtained by Corollary 1.20. Any invertible bounded analytic functions must be an outer function, for example,

$$g(z) = z - \alpha$$

is outer for every $|\alpha| > 1$. In fact it can be checked that when $|\alpha| = 1$, g is still an outer function since in this case $g \in H^1$ and $1/g \in H^p$ for every $p \in (0, 1)$.

It is evident from Corollary 1.20 that the product of finitely many *bounded* or H^2 outer functions is again outer, therefore

$$\prod_{i=1}^{n} (z - \alpha_i)$$

is an outer function whenever $|\alpha_i| \geq 1$. ∎

Finer properties of inner and outer functions are in order: Inner functions satisfy $|f(z)| \leq 1$ on $\mathbb{D}$ by Corollary 1.12. Also, suppose that $F(z)$ is an outer function with the same notations used in Definition 1.14, then $F(z) \in H^p$ since it is analytic and by the fact $|F(z)|^p = \exp\{pP[\ln \varphi](z)\}$ and by Jensen's inequality, we have for every $0 \leq r < 1$,

$$\begin{aligned}
\|F_r\|_p^p &= \int_T |F_r(z)|^p \, \frac{|dz|}{2\pi} \\
&\leq \int_T P[\varphi^p]_r(z) \, \frac{|dz|}{2\pi} \\
&= \|P[\varphi^p]_r\|_1 \\
&\leq \|\varphi^p\|_1 \\
&= \|\varphi\|_p^p,
\end{aligned}$$

where the last inequality follows from Theorem 1.7, therefore (1.2) holds. From Theorem 1.5, $F(z)$ has nontangential limit $F(t)$ a.e.. Since $P[\ln \varphi]_r \to \ln \varphi$ in

L^1, for a.e. t,

$$|F(t)|^p = \exp\{p \ln \varphi(t)\} = \varphi^p(t).$$

Therefore the class of outer functions for the class H^p is a subclass of H^p functions that can be uniquely determined by their magnitude on the boundary up to a unimodular constant:

$$F(z) = e^{i\gamma} \exp\left\{\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \ln|F(t)| \frac{dt}{2\pi}\right\}$$

with $\ln|F(t)| \in L^1$ and $|F(t)| \in L^p$. In particular,

$$\ln|F(0)| = \int_T \ln|F(t)| \frac{dt}{2\pi},$$

and this turns out to be also sufficient for a H^1 function to be outer (see Theorem 1.18 below).

Theorem 1.17 ([RUD], p. 342). *A singular inner function is of the form*

$$\exp\left\{-\int_T \frac{\zeta+z}{\zeta-z} \, d\mu(\zeta)\right\}$$

for some unique positive singular measure on T. Conversely, any analytic function of this form is also a singular inner function.

We will use the following useful characterization of outer functions:

Theorem 1.18 ([GAR], p. 64). *Let $0 < p \leq \infty$ and let $f \in H^p$, $f \not\equiv 0$, then the following are equivalent.*

(a) $f(z)$ is an outer function.

(b) For each $z \in \mathbb{D}$, equality (1.11) holds, i.e.,

$$\ln|f(z)| = \int_0^{2\pi} \ln|f(t)| P_z(t) \frac{dt}{2\pi}. \tag{1.19}$$

(c) For some $z_0 \in \mathbb{D}$ (1.19) holds.

(d) For every $g \in H^p$ such that $|g(t)| = |f(t)|$ a.e., then

$$|g(z)| \leq |f(z)| \quad \textit{for all } z \in \mathbb{D}.$$

Corollary 1.20. *If $f \in H^p$ and if for some $r > 0$, $1/f \in H^r$, then $f(z)$ is an outer function.*

Theorem 1.21 ([GAR], p. 71). *If $p > 0$ and $f \in H^p$, then $f(z)$ has a unique decomposition*

$$f(z) = CB(z)S(z)F(z),$$

where $|C| = 1$, $B(z)$ is a Blaschke product, $S(z)$ is a singular inner function and $F(z)$ is an outer function in H^p.

1.3 Introduction to Interpolating Sequences

In this section we mention all results that we need and present the surprisingly simple proof of P. Jones on explicit construction to interpolation problem. This proof is very instructive that certain detail can be extracted to give an original and constructive proof to an interpolation problems (Theorem 2.11) in which the important condition stated in Definition 1.23 fails.

Interpolating sequence plays an important role in the study of bounded analytic functions. For example, they are used in Chapter 9 and 10 of Garnett's [GAR] on characterization of closed algebras between H^∞ and L^∞, and also on the discussion of maximal ideal space.

1.3.1 Pseudohypoerbolic Distance

The following metric on $\mathbb{D}$ turns out to be more natural than the Euclidean distance for problems in the open unit disk.

Definition 1.22. The *pseudohyperbolic distance* of $a, b \in \mathbb{D}$ is defined by

$$\rho(a, b) = \left| \frac{a - b}{1 - \overline{a}b} \right|.$$

Let's list a few properties of ρ which we use later. Firstly, we will denote

$$\varphi_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}, \quad \alpha \in \mathbb{D}.$$

With this notation, we have $\varphi_\alpha \circ \varphi_{-\alpha} = \mathrm{id}$ and $\rho(\alpha, z) = |\varphi_\alpha(z)|$. We know that:

- For any analytic $\tau : \mathbb{D} \to \overline{\mathbb{D}}$, we have

$$\rho(\tau(z), \tau(\zeta)) \leq \rho(z, \zeta),$$

  known as the generalized Schwarz Lemma. With equality holds for some z, ζ if and only if τ is of the form $c\varphi_\alpha$ for some $|c| = 1$ and $\alpha \in \mathbb{D}$, and in that case the equality must be attained thoroughly for all $z, \zeta \in \mathbb{D}$, i.e., φ_α is an isometry w.r.t. pseudohyperbolic distance.

- The quantities $1 - |\alpha|^2, 1 - |z|^2$ and $1 - |\rho(\alpha, z)|^2$ are related by

$$1 - |\rho(\alpha, z)|^2 = \frac{(1-|\alpha|^2)(1-|z|^2)}{|1-\overline{\alpha}z|^2}.$$

1.3.2 Separation and Uniform Separation Condition

Definition 1.23. A sequence $\{z_n\}$ in $\mathbb{D}$ is *interpolating* if for every bounded sequence $\{a_n\}$ in $\mathbb{C}$, we can find an $f \in H^\infty$ such that

$$f(z_j) = a_j.$$

We note that interpolating sequence cannot be arbitrary. For example, suppose $\{z_n\}$ is an interpolating sequence and set $a_1 = 1$, $a_2 = a_3 = \cdots = 0$, then there is a bounded analytic $f(z)$ such that $f(z_1) = 1$ and $f(z_2) = f(z_3) = \cdots = 0$. Since $z_2, z_3, \ldots$ are zeros of $f(z)$ and $f(z)$ is nonzero, $\{z_n\}$ must satisfy

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty. \tag{1.24}$$

A short reason is as follows. W.l.o.g. we assume $|f| \leq 1$ and $f(0) \neq 0$. If we set $B_n = \prod_{k=2}^{n} \frac{z-z_k}{1-\overline{z_k}z}$, then $|f/B_n| \leq 1$ on T, hence on $\mathbb{D}$, and thus if we set $z = 0$,

$$0 < |f(0)| \leq |B_n(0)| = \prod_{k=2}^{n} |z_n|.$$

As this holds for every n, $\prod |z_n| > 0$, therefore $\sum(1 - |z_n|) < \infty$, as desired.

This kind of sequences is so important that they bear a name:

Definition 1.25. A sequence $\{z_n\}$ in $\mathbb{D}$ satisfying (1.24) is called a *Blaschke sequence.*

Let's continue to assume $\{z_n\}$ is interpolating in $\mathbb{D}$. We try to seek for its basic properties beyond those of a Blaschke sequence. Define

$$T : H^\infty \to \ell^\infty; \quad f \mapsto (f(z_1), f(z_2), \dots).$$

By definition T is a surjective bounded operator, therefore by Open Mapping Theorem there is a constant $M > 0$ such that for every $a \in \ell^\infty$, there is a solution $f \in H^\infty$ such that

$$\|f\|_\infty \leq M\|a\|_\infty. \tag{1.26}$$

The smallest possible constant M that satisfies this inequality is

$$M = \sup_{\|\{a_j\}\|_\infty \leq 1} \min\{\|f\|_\infty : f \in H^\infty, f(z_j) = a_j, j = 1, 2, \dots\}. \tag{1.27}$$

The minimum in the above quantity exists by a normal family argument. It is easy to verify (1.27) indeed satisfies (1.26) for those solution $f(z)$ of minimal norm. Moreover, this M is independent of the choice of $\{a_n\}$ (but depends on $\{z_n\}$).

Definition 1.28. The constant in (1.27) is called the *constant of interpolation.*

Now we try to investigate geometric properties of the interpolating sequence $\{z_n\}$. Let M be the constant of interpolation. For every $k \in \mathbb{N}$, let $a_j = \delta_{kj}$ for $j = 1, 2, \dots$, then we can find an $f \in H^\infty$ such that

$$f(z_j) = \delta_{jk} \quad \text{and} \quad \|f\|_\infty \leq M\|\{a_n\}\|_\infty = M,$$

it follows from generalized Schwarz Lemma that for every $j \neq k$, the pseduohyperbolic distance $\rho(z_j, z_k)$ is bounded below:

$$\left|\frac{z_j - z_k}{1 - \overline{z_k} z_j}\right| = \rho(z_j, z_k) \geq \rho\left(\frac{f(z_j)}{M}, \frac{f(z_k)}{M}\right) = \left|\frac{0 - \frac{1}{M}}{1 - 0}\right| = \frac{1}{M}. \tag{1.29}$$

This motivates our next definition.

Definition 1.30. A sequence $\{z_n\}$ in $\mathbb{D}$ is *separated* if there is an $a > 0$ such that

$$\left|\frac{z_j - z_k}{1 - \overline{z_k} z_j}\right| \geq a, \quad \text{for every } j, k \in \mathbb{N}, \quad j \neq k.$$

We have just shown that every interpolating sequence is separated in (1.29). Moreover, if we let B be the Blaschke product with zeros z_j $(j \neq k)$, i.e.,

$$B(z) = \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \frac{-\overline{z_j}}{|z_j|} \frac{z - z_j}{1 - \overline{z_j} z},$$

then since z_j's $(j \neq k)$ are the zeros of f, f/B can be regarded as a nonvanishing bounded analytic function and

$$1 = f(z_k) = B(z_k)\frac{f(z_k)}{B(z_k)} \leq |B(z_k)| \left\|\frac{f}{B}\right\|_{\infty} = |B(z_k)| \|f\|_{\infty} \leq |B(z_k)| M,$$

thus we have for each k,

$$\prod_{j \neq k} \left|\frac{z_k - z_j}{1 - \overline{z_j} z_k}\right| = |B(z_k)| \geq \frac{1}{M} > 0.$$

This condition is much stronger than being separated and we call:

Definition 1.31. A sequence $\{z_n\}$ is *uniformly separated* if

$$\inf_{k \geq 1} \prod_{j \neq k} \left|\frac{z_k - z_j}{1 - \overline{z_j} z_k}\right| > 0.$$

In 1958 Carleson proved that the uniform separation condition is also sufficient for a sequence to be interpolating in Theorem 1.33. Not only that, part (c) of this theorem provides a more geometric condition which we will use later in the proof of Theorem 2.1. The statement of Theorem 1.33 involves a special kind of annular sector that will also be used in a proof of our main result in the next chapter:

Definition 1.32. Any annular sector in $\mathbb{D}$ of the form

$$Q = \{re^{i\theta} : 1 - h \leq r < 1, |\theta - \theta_0| \leq h\}$$

is called *Carleson square*, and we denote h by $\ell(Q)$.

Figure 1.1 gives the general picture of Carleson squares.

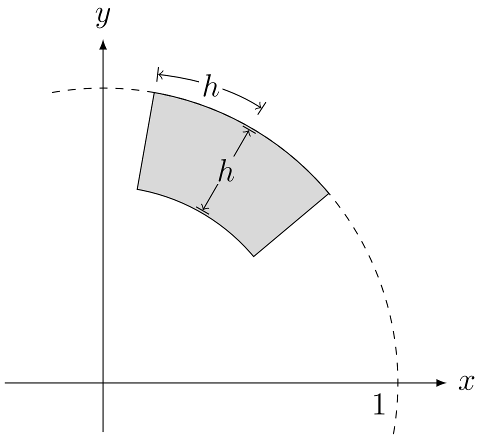

Figure 1.1: A Carleson square.

Theorem 1.33 (Carleson). *Let $\{z_n\}$ be a sequence in $\mathbb{D}$, the following conditions are equivalent:*

(a) The sequence is an interpolating sequence.

(b) We have

$$\delta = \inf_{k\geq 1} \prod_{n\neq k} \left| \frac{z_k - z_n}{1 - \overline{z_n} z_k} \right| > 0.$$

(c) The points z_n's are separated: There is $a > 0$ such that

$$\rho(z_j, z_k) := \left| \frac{z_j - z_k}{1 - \overline{z_k} z_j} \right| \geq a > 0$$

for every j, k distinct; and there is an absolute constant C such that for every square $Q = \{re^{i\theta} : 1 - \ell(Q) \leq r < 1, |\theta - \theta_0| \leq \ell(Q)\}$,

$$\sum_{z_n \in Q} (1 - |z_n|) \leq C\ell(Q). \tag{1.34}$$

The constant δ above and the constant of interpolation

$$M = \sup_{\|a_j\|_\infty \leq 1} \inf\{\|f\|_\infty : f(z_j) = a_j, j = 1, 2, \ldots, f \in H^\infty\}$$

are related by the inequalities

$$\frac{1}{\delta} \leq M \leq \frac{c}{\delta}\left(1 + \ln\frac{1}{\delta}\right), \tag{1.35}$$

in which c is some absolute constant.

As the open unit disk and the upper half plane

$$\mathbb{H} := \{z : \operatorname{Im} z > 0\}$$

are conformally equivalent, the interpolation at sequences in $\mathbb{D}$ is the same as interpolation at sequences in $\mathbb{H}$. It is not surprising we have an extremely similar analogue of Theorem 1.33 in $\mathbb{H}$, for a completely self-contained proof, see p. 278 of [GAR]. One can also find the proof in Chapter 9 of [KOO] which relies heavily on facts related to Carleson measures (to be defined in Definition 1.37) with many motivating pictures.

The estimate (1.35) is sharp, for an example, see p. 284 of [GAR].

Note that in part (c) of Theorem 1.33 the sum $\sum_{z_n \in Q}(1-|z_n|)$ can be rewritten as

$$\left(\sum_{n=1}^{\infty}(1-|z_n|)\delta_{z_n}\right)(Q).$$

Where δ_{z_n} denotes the Dirac measure at z_n. If we denote $\mu = \sum_{n=1}^{\infty}(1-|z_n|)\delta_{z_n}$, then (1.34) can be written as

$$\mu(Q) \leq C\ell(Q). \tag{1.36}$$

This also bears a name and it is found to be useful elsewhere.

Definition 1.37. Any Borel measure μ on $\mathbb{D}$ such that there is a constant $C > 0$ for which (1.36) holds for every Carleson square Q is called a *Carleson measure.*

1.3.3 An Example and a Nonexample of Interpolating Sequences

Since it is difficult both to understand and to verify the uniform separation condition, the existence of interpolating sequence is not so obvious to us at the moment. Fortunately the uniform separation condition has a simple sufficient growth condition:

Theorem 1.38 ([DUR], p. 155). *If there is a constant* $c \in (0,1)$ *such that*

$$1 - |z_{n+1}| \leq c(1 - |z_n|) \quad \textit{for every } n, \tag{1.39}$$

then $\{z_n\}$ *is uniformly separated. Moreover, the condition (1.39) is also necessary if* $0 \leq z_1 < z_2 < \cdots$.

From this it is easy to raise the following examples:

Example 1.40 (Interpolating Sequence). Let $z_1, z_2, \cdots > 0$ and take $z_1 = \frac{1}{2}$. Replacing $\leq$ by $=$ in (1.39), for any $c \in (0,1)$ we obtain an interpolating sequence $\{z_n\}$ given by $z_n = 1 - \dfrac{c^{n-1}}{2}$. ∎

On the other hand, there is a Blaschke sequence that is not interpolating:

Example 1.41 (Noninterpolating Sequence). We take $z_n = 1 - \dfrac{1}{n^2}$. This sequence is increasing, but if (1.39) holds for some $c \in (0,1)$, then we have

$$\frac{(n+1)^2 - 1}{(n+1)^2} \cdot \frac{n^2}{n^2 - 1} \leq c$$

for every n, forcing $c \geq 1$, impossible, so $\{1 - \frac{1}{n^2}\}$ can't be interpolating.

For more examples, of course any sequence containing a noninterpolating sequence must not be interpolating either. ∎

1.3.4 The Simple Constructive Proof to Existence of Interpolation by P. Jones

P. Jones gave an extremely simple proof to Carleson Theorem in [JON], formulated in the form of Theorem 1.42. This proof is stunning in his time because of Carleson's work and because very few people consider the explicit construction possible.

Theorem 1.42. *Let* $\{\lambda_n\}$ *be a uniformly separated sequence such that* $0 <$

$|\lambda_n| \le |\lambda_{n+1}|$ *for* $n = 1, 2, 3, \ldots$. *Denote*

$$\delta = \inf_{n\ge 1} \prod_{j\ne n} \left|\frac{\lambda_j - \lambda_n}{1-\overline{\lambda_n}\lambda_j}\right| > 0, \quad B_n(z) = \prod_{\substack{j=1\\ j\ne n}}^{\infty} \frac{-\overline{\lambda_j}}{|\lambda_j|}\frac{z-\lambda_j}{1-\overline{\lambda_j}z}$$

and also

$$\alpha_n(z) = \sum_{k\ge n} \frac{1+\overline{\lambda_k}z}{1-\overline{\lambda_k}z}(1-|\lambda_k|^2).$$

Finally, let

$$\Phi_n(z) = \left(\frac{1-|\lambda_n|^2}{1-\overline{\lambda_n}z}\right)^2 \frac{B_n(z)}{B_n(\lambda_n)} \exp\big(\varepsilon(\alpha_n(\lambda_n) - \alpha_n(z))\big), \tag{1.43}$$

where $\varepsilon = (2\ln\frac{e}{\delta^2})^{-1}$, *then*

$$\sum_{n=1}^{\infty} |\Phi_n(z)| \le \frac{2e}{\delta}\ln\frac{e}{\delta^2}, \quad \text{for } z \in \mathbb{D}.$$

As a result, these Φ_n's directly gives an interpolation for every bounded sequence at points $\{z_n\}$ by setting $f = \sum a_n \Phi_n$.

Proof. This follows from very careful computation. First of all, we have

$$\mathrm{Re}(\alpha_n(z)) = \sum_{k\ge n} \frac{1-|z|^2|\lambda_k|^2}{|1-\overline{\lambda_k}z|^2}(1-|\lambda_k|^2). \tag{1.44}$$

For every $k \ge n$, we have $|\lambda_k| \ge |\lambda_n|$, therefore $(1-|\lambda_n|^2)(1+|\lambda_k|^2) \ge 1 - |\lambda_n|^2|\lambda_k|^2$, thus

$$\begin{aligned}
\mathrm{Re}\,\alpha_n(\lambda_n) &\le \sum_{k\ge n} \frac{(1-|\lambda_k|^4)(1-|\lambda_n|^2)}{|1-\overline{\lambda_k}\lambda_n|^2} \\
&\le 2\sum_{k\ge n} \frac{(1-|\lambda_k|^2)(1-|\lambda_n|^2)}{|1-\overline{\lambda_k}\lambda_n|^2} \\
&= 2\sum_{k\ge n}(1-\rho(\lambda_k,\lambda_n)^2) \\
&= 2\left(1+\sum_{k>n}(1-\rho(\lambda_k,\lambda_n)^2)\right) \\
&\le 2\left(1-\sum_{k>n}\ln\rho(\lambda_k,\lambda_n)^2\right) \le 2(1-\ln\delta^2) =: \frac{1}{\varepsilon}.
\end{aligned} \tag{1.45}$$

By (1.45) and the definition of Φ_n in (1.43) we have

$$
\begin{aligned}
|\Phi_n(z)| &\leq \frac{(1-|\lambda_n|^2|z|^2)}{|1-\overline{\lambda_n}z|^2}(1-|\lambda_n|^2)\cdot\frac{1}{\delta}\cdot\exp(1-\varepsilon\operatorname{Re}\alpha_n(z)) \\
&= \frac{e}{\delta}(\operatorname{Re}\alpha_n(z)-\operatorname{Re}\alpha_{n+1}(z))\exp(-\varepsilon\operatorname{Re}\alpha_n(z)) \\
&\leq \frac{e}{\varepsilon\delta}(e^{\varepsilon\operatorname{Re}\alpha_n(z)-\varepsilon\operatorname{Re}\alpha_{n+1}(z)}-1)e^{-\varepsilon\operatorname{Re}\alpha_n(z)} \\
&= \frac{e}{\varepsilon\delta}(e^{-\varepsilon\operatorname{Re}\alpha_{n+1}(z)}-e^{-\varepsilon\operatorname{Re}\alpha_n(z)}).
\end{aligned}
\tag{1.46}
$$

Finally we take summation on both sides to get $\sum_{n=1}^{\infty}|\Phi_n(z)|\leq e/(\varepsilon\delta)$. ∎

The key to the proof is to write the expression

$$\frac{(1-|\lambda_n|^2|z|^2)}{|1-\overline{\lambda_n}z|^2}(1-|\lambda_n|^2)$$

in (1.46) as a difference of two "adjacent" numbers in a sequence. In [VIN] Vinogradov found a similar interpolation operator for

$$T_{\{z_n\}}: H^p\to\ell^p;\quad f\mapsto(f(z_1),f(z_2),\dots).$$

Theorem 1.47. *Suppose $\{\lambda_n\}$ is uniformly separated. Let*

$$\delta=\inf_{n\geq 1}\prod_{j\neq n}\rho(\lambda_j,\lambda_n)$$

and let

$$|\lambda_1|\leq|\lambda_2|\leq\cdots.$$

Denote

$$b_0(z)=z,\quad b_\lambda(z)=\frac{|\lambda|}{\lambda}\frac{\lambda-z}{1-\overline{\lambda}z}\quad\text{and}\quad a_\lambda(z)=\frac{\sqrt{|\lambda|}}{\lambda}\frac{\lambda-|\lambda|z}{1-\overline{\lambda}z}$$

for $z\in\widehat{\mathbb{C}}$ and $\lambda\in\mathbb{D}\setminus\{0\}$. Also let

$$A_n=\prod_{k>n}a_{\lambda_k},$$

then

$$\sum_{n=1}^{\infty}\left(\frac{1-|\lambda_n|^2}{|1-\overline{\lambda_n}z|}\right)^2|A_{\lambda_n}(z)|\leq 8$$

and the interpolation is given by

$$Q(z)=\sum_{n=1}^{\infty}a_n\left(\frac{1-|\lambda_n|^2}{1-\overline{\lambda_n}z}\right)^2\frac{B_n(z)}{B_n(\lambda_n)}\frac{A_{\lambda_n}(z)}{A_{\lambda_n}(\lambda_n)}\tag{1.48}$$

with $\|Q\|_p\leq(8/\delta^2)\|a\|_\infty$.

(1.43) and (1.48) suggest a strong evidence that explicit construction look quite the same. Namely, the term

$$\left(\frac{1-|\lambda_n|^2}{1-\overline{\lambda_n}z}\right)^2 \frac{B_n(z)}{B_n(\lambda_n)}$$

must be there! They motivate the construction in (2.13) in the next chapter, where C_k is to be determined. We will adopt the similar idea as in Jones's proof above to push certain telescoping expression to appear in the proof of Theorem 2.11 later.

Chapter 2

Interpolating Sequences

2.1 Uniformly Separated Sequences

We have seen that interpolating sequence must be a Blaschke sequence in Section 1.3.2. On the other hand, Section 1.3.3 shows us not every Blaschke sequence is interpolating. The next theorem states that interpolating sequence can have arbitrary growth rate as a Blaschke sequence!

Theorem 2.1. *If $|z_j| < 1$ for every $j \geq 1$ and if $\sum_{j=1}^{\infty}(1 - |z_j|) < \infty$, then there is an interpolating sequence $\{w_j\}$ with $|w_j| = |z_j|$ for every $j \geq 1$.*

This result is cited in p. 305 of [GAR] and was due to Naftalevitch in the 1940's. However, the search for Naftalevitch's original paper containing this result is not successful. We try to fill in the proof on our own:

Proof. We may assume $|z_1| \leq |z_2| \leq \cdots$. The proof is constructive. We define

$$\theta_k = \sum_{i \geq k}(1 - |z_i|),$$

and we hope that

$$w_k = |z_k| e^{i\theta_k}, \quad k = 1, 2, \ldots \tag{2.2}$$

form an interpolating sequence. By part (c) of Theorem 1.33 a sequence $\{w_n\}$ is interpolating if and only if $\{w_n\}$ is separated and the discrete measure

$$\mu = \sum_{n=1}^{\infty}(1 - |w_n|)\delta_{w_n}$$

is Carleson. That is, we need to prove the following separation condition

$$\inf\left\{\rho(w_j, w_k) := \left|\frac{w_j - w_k}{1 - \overline{w_j}w_k}\right| : j \neq k\right\} > 0 \tag{2.3}$$

and for every Carleson square

$$Q = \{re^{i\theta} \in \mathbb{D} : 1 - h \leq r < 1, |\theta - \theta_0| \leq h\} =: C(\theta, h), \quad h > 0,$$

we have the Carleson condition

$$\mu(Q) \leq C\ell(Q) \tag{2.4}$$

(where $\ell(Q) := h$) for some absolute constant $C > 0$.

We first prove the Carleson condition (2.4), we will then prove that $\{w_{2n}\}$ is separated and conclude that we don't need to show $\{w_n\}$ itself is separated.

Proof of Carleson condition (2.4). We consider three types of Carleson squares:

Case 1. Consider $Q \subseteq \mathbb{H}$ such that $1 \notin Q$. Let $Q := C(\phi, h)$ be such that $\phi - h > 0$ (i.e., $1 \notin Q$). We give a sketch of those Q's in Figure 2.1.

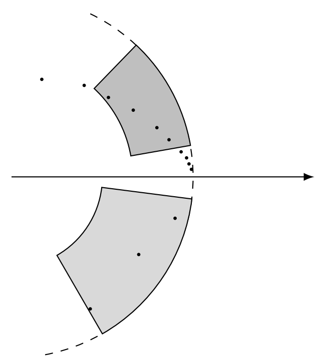

Figure 2.1: Carleson squares not containing 1.

Since $\{w_k\}$ defined in (2.1) only accumulates at 1, we may let k be the *first* integer such that $w_k \in Q$ and n the *last* integer such that $w_n \in Q$, then

$$\phi - h \leq \theta_n \leq \theta_k \leq \phi + h,$$

and therefore

$$\mu(Q) \le \sum_{k \le i \le n} (1 - |w_i|) = \theta_k - \theta_n + (1 - |w_n|) \le (\phi + h) - (\phi - h) + h = 3h.$$

The case for those Q's in the lower half plane is essentially the same.

Case 2. In the remaining two cases let's consider Carleson squares containing 1. First consider those centered at 1:

$$Q = C(0, h).$$

Let k_0 be the first integer such that $w_{k_0} \in Q$, then by definition,

$$\sum_{i \ge k_0} (1 - |w_i|) = \theta_{k_0} \le h.$$

Now for every $k \ge k_0$, we have $1 - |w_k| \le h$, i.e., $1 - h \le |w_k|$. We also have $\theta_k \le \theta_{k_0} \le h$. Therefore $w_k \in C(0, h)$. So we have

$$\mu(Q) = \sum_{i \ge k_0} (1 - |w_i|) \le h.$$

Case 3. Finally for every Carleson square Q such that $1 \in Q$, let $h = \ell(Q)$, then it must happen that $Q \subseteq C(0, 2h)$, as shown in Figure 2.2.

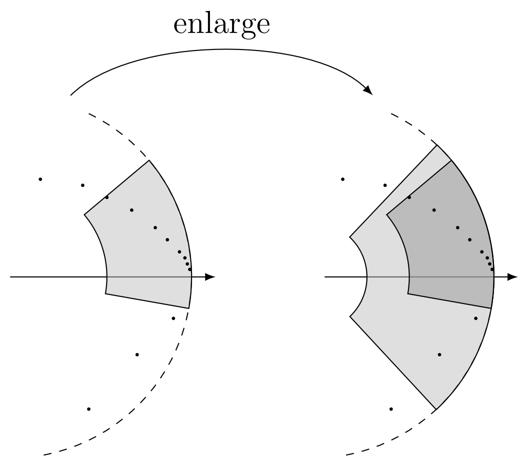

Figure 2.2: Extension to Carleson squares centered at 1.

Therefore by case 2,

$$\mu(Q) \le \mu(C(0, 2h)) \le 2h.$$

Combining 3 cases, (2.4) holds with $C = 3$.

Proof of Separation Condition (2.3). We try to shrink

$$\rho(w_j, w_k)^2 = 1 - \frac{(1-|w_j|^2)(1-|w_k|^2)}{|1-\overline{w_k}w_j|^2}.$$

Since $\{w_n\}$ accumulates only at 1, we may just show that $\rho(w_j, w_k)$ is bounded below for sufficiently big j and k.

We first shrink the factor $|1-\overline{w_k}w_j|$. We may assume $j < k$, then $\theta_j - \theta_k > 0$. Let $\epsilon \in (0, 1)$ be very small and let k, j be sufficiently large.

$$\begin{aligned} |1-\overline{w_k}w_j|^2 &= |1-|z_k||z_j|e^{i(\theta_j-\theta_k)}|^2 \\ &= (1-|z_k||z_j|\cos(\theta_j-\theta_k))^2 + |z_k|^2|z_j|^2\sin^2(\theta_j-\theta_k), \end{aligned}$$

set $\boxed{a = |z_j|}$ and $\boxed{b = |z_k|,}$ then the above

$$= a^2b^2 + 1 - 2ab\cos(\theta_j - \theta_k). \tag{2.5}$$

Now we divide the proof into two steps.

Step 1. Assume that $k - j \geq 2$, then

$$\begin{aligned} \theta_j - \theta_k &= \sum_{i=j}^{k-1}(1-|z_i|) \\ &\geq 1-|z_j| + 1 - |z_{k-1}| \\ &\geq 1-|z_j| + 1 - |z_k| \geq 2\sqrt{(1-|z_j|)(1-|z_k|)}, \end{aligned}$$

and hence

$$(\theta_j - \theta_k)^2 \geq 4(1-|z_j|)(1-|z_k|) = 4(1-a)(1-b) =: B. \tag{2.6}$$

Since for $\epsilon > 0$, $\cos x < 1 - (\frac{1}{2} - \epsilon)x^2$ for sufficiently small x, it follows that from (2.5) and (2.6),

$$\begin{aligned} |1-\overline{w_k}w_j|^2 &\geq a^2b^2 + 1 - 2ab\left(1 - (\frac{1}{2} - \epsilon)B\right) \\ &= a^2b^2 + 1 - 2ab + ab(1-2\epsilon)B \\ &= (1-ab)^2 + abB - 2abB\epsilon, \end{aligned}$$

recalling $B = 4(1-a)(1-b)$ and observing that $(1-ab)^2 - (1-a)(a-b) = a^2b^2 + a + b - 3ab \geq 0$, we conclude the above

$$\begin{aligned}
&\geq (1-a)(1-b) + ab[4(1-a)(1-b)] - 8(1-a)(1-b)\epsilon \\
&= (1-a)(1-b)[1 + 4ab - 8\epsilon] \\
&= (1-|w_j|)(1-|w_k|)(1+4|w_j||w_k| - 8\epsilon). \qquad (2.7)
\end{aligned}$$

Choose $\epsilon = \frac{1}{32}$ at the beginning, then for j, k large, $1+4|w_j||w_k| > 4.75$, therefore from (2.7),

$$|1 - \overline{w_k}w_j|^2 \geq 4.5(1-|w_j|)(1-|w_k|),$$

and we have

$$\begin{aligned}
\rho(w_j, w_k)^2 &= 1 - \frac{(1-|w_j|^2)(1-|w_k|^2)}{|1-\overline{w_k}w_j|^2} \\
&\geq 1 - \frac{4(1-|w_j|)(1-|w_k|)}{4.5(1-|w_j|)(1-|w_k|)} = \frac{1}{9}.
\end{aligned}$$

Step 2. Suppose now $k = j+1$, the same technique does not work since $\theta_j - \theta_k = 1 - |z_j|$ is not a sum of at least two terms in $\{1 - |z_i| : i \geq 1\}$. Fortunately we don't need to do this step! By Step 1, given any Blaschke sequence $|z_n|$, our construction shows that $\{w_{2n}\}$ is separated, and thus interpolating since previously we have shown that $\sum(1-|w_{2n}|)\delta_{w_{2n}}$ is Carleson, with $|w_{2n}| = |z_{2n}|$. From this point, given any Blaschke sequence $z_1, z_2, \ldots$, construct another Blaschke sequence

$$a_1, z_1, a_2, z_2, \ldots,$$

such that $|a_1| \leq |z_1| \leq |a_2| \leq |z_2| \leq \cdots$, by Step 1 we get an interpolating sequence $\{w_n'\}$ with $|w_{2n}'| = |z_n|$. ∎

2.2 Nonuniformly Separated Sequences

Let $\lambda_1, \lambda_2, \cdots \in \mathbb{D}$. Recall that $\{\lambda_n\}$ is said to be *uniformly separated* if

$$\delta_n := \prod_{\substack{k=1 \\ k\neq n}}^{\infty} \left| \frac{\lambda_n - \lambda_k}{1 - \overline{\lambda_k}\lambda_n} \right| > 0, \quad n = 1, 2, 3, \ldots$$

satisfy $\inf_{n\geq 1}\delta_n > 0$. In case

$$\inf_{n\geq 1}\delta_n = 0,$$

Carleson's Theorem fails. Nevertheless, under a suitable growth condition on the sequence of numbers $\{a_n\}$ (rather than just a bounded sequence) a positive result was found by Garnett:

Theorem 2.8 ([GAR2]). *Let $A(t)$ be a positive decreasing function on $[0,\infty)$. If*

$$\int_0^\infty A(t)\,dt < \infty$$

and if

$$|a_n| \leq \delta_n A(1+\ln 1/\delta_n) \quad \textit{for each } n, \tag{2.9}$$

then there is an absolute constant $C>0$ for which there is an interpolation

$$f \in H^\infty, f(\lambda_n) = a_n, \quad n = 1,2,\ldots$$

such that

$$\|f\|_\infty \leq C\int_0^\infty A(t)\,dt. \tag{2.10}$$

In case $\inf\delta_n > 0$, Theorem 2.8 also includes Theorem 1.33 as a special case by taking $A(t) = \chi_{[0,1+\ln 1/\delta_n]}(t)$.

The proof of Theorem 2.8, i.e., Theorem 4 in [GAR2], is done in the upper half plane, and the existence of the solution rests on showing certain discrete measure is Carleson in an attempt to give an upper bound of

$$M_n := \inf\{\|f\|_\infty : f\in H^\infty, f(z_j) = a_j, 1\leq j\leq n\}$$

that is independent of n. A similar result can be proved constructively when the growth condition (2.9) is modified to (2.12).

Theorem 2.11. *Let $A(t)$ be a positive decreasing function on $[0,\infty)$. If*

$$\int_0^\infty A(t)\,dt < \infty,$$

$\{t_n\}$ diverges to ∞ and if there is $p\geq 2$ such that

$$|a_n| \leq \delta_n^p A(t_n) \quad \textit{for each } n, \tag{2.12}$$

then there is an absolute constant $C > 0$ for which there is an interpolation

$$f \in H^\infty, f(\lambda_n) = a_n, \quad n = 1, 2, \dots$$

such that

$$\|f\|_\infty \leq C \int_0^\infty A(t)\, dt.$$

When $t_n = 1 + \ln 1/\delta_n$, then this becomes a weakened version of Theorem 2.8. Our modification is flexible in the sense that t_n can be arbitrary, as long as it diverges to ∞, rather than using the fixed choice $\{1 + \ln 1/\delta_n\}$.

The special case that $\inf \delta_n > 0$ in our Theorem 2.11 still generalizes Theorem 1.33. The use of the condition $p \geq 2$ will be seen in line (2.20) in which we try to shrink a number $|a|$ to $|a|^{p-1}$ with $|a| < 1$.

Proof of Theorem 2.11. W.l.o.g. let's assume $|\lambda_1| \leq |\lambda_2| \leq \cdots$. We try to construct an explicit construction as P. Jones did in [JON]. We define a similar sum that's supposed to be a correct interpolation

$$S(z) = \sum_{n=1}^{\infty} a_n \left(\frac{1 - |\lambda_n|^2}{1 - \overline{\lambda_n} z} \right)^2 \frac{B_n(z)}{B_n(\lambda_n)} \cdot C_n(z), \tag{2.13}$$

here $C_n(z)$ is to be determined such that $C_n(\lambda_n) = 1$ and

$$B_n(z) = \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \left(\frac{-\overline{\lambda_k}}{|\lambda_k|} \right) \frac{z - \lambda_k}{1 - \overline{\lambda_k} z}.$$

Supposing the summation converges normally by suitably chosen $C_n(z)$'s, then we have

$$\begin{aligned}
|S(z)| &\leq \sum_{n=1}^{\infty} |a_n| \frac{(1 - |\lambda_n|^2)^2}{|1 - \overline{\lambda_n} z|^2} \left| \frac{B_n(z)}{B_n(\lambda_n)} \right| |C_n(z)| \\
&\leq \sum_{n=1}^{\infty} [\delta_n^p A(t_n)] \frac{(1 - |\lambda_n|^2 |z|^2)(1 - |\lambda_n|^2)}{|1 - \overline{\lambda_n} z|^2} \frac{1}{\delta_n} |C_n(z)| \\
&= \sum_{n=1}^{\infty} A(t_n) \frac{(1 - |\lambda_n|^2 |z|^2)(1 - |\lambda_n|^2)}{|1 - \overline{\lambda_n} z|^2} |C_n(z)| \delta_n^{p-1}. \qquad (2.14)
\end{aligned}$$

From [VIN] we have the following computation: define

$$a_{\lambda_n}(z) = \frac{\sqrt{|\lambda_n|}}{\lambda_n} \left(\frac{\lambda_n - |\lambda_n| z}{1 - \overline{\lambda_n} z} \right),$$

then

$$1 \geq |a_{\lambda_n}(z)|^2 = 1 - \frac{(1-|\lambda_n||z|^2)(1-|\lambda_n|)}{|1-\overline{\lambda_n}z|^2}.$$

Therefore the line (2.14) becomes

$$|S(z)| \leq 4\sum_{n=1}^{\infty} A(t_n)(1-|a_{\lambda_n}(z)|^2)|C_n(z)|\delta_n^{p-1}. \tag{2.15}$$

Now we follow Garnett's proof in [GAR2] to consider the following sets of intergers

$$E_m = \{k \in \mathbb{N} : m-1 \leq t_k < m\},$$

then from (2.15) we get

$$|S(z)| \leq 4\sum_{m=1}^{\infty}\sum_{k\in E_m}(\cdots) \leq 4\sum_{m=1}^{\infty}\underbrace{\left(\sum_{k\in E_m}(1-|a_{\lambda_k}(z)|^2)|C_k(z)|\delta_k^{p-1}\right)}_{:=B_m}A(m-1). \tag{2.16}$$

Next we perform summation by parts once to (2.16), recall that

$$\sum_{m=1}^{N} a_m b_m = \sum_{m=1}^{N-1}(a_1+\cdots+a_m)(b_m-b_{m+1}) + (a_1+\cdots+a_N)b_N.$$

Putting $a_m = B_m$ and $b_m = A(m-1)$ we have from (2.16) that

$$|S(z)| \leq 4\lim_{N\to\infty}\left(\begin{array}{c}\sum_{m=1}^{N-1}(B_1+\cdots+B_m)(A(m-1)-A(m)) \\ +(B_1+\cdots+B_N)A(N-1)\end{array}\right). \tag{2.17}$$

Now we aim to show that for suitably chosen $C_n(z)$ we have for some absolute constant $C>0$ such that

$$B_1+\cdots+B_m \leq Cm \quad \text{for every } m. \tag{2.18}$$

Let's accept (2.18) for the moment and finish the proof quickly. Since $A(t)$ is decreasing and integrable, we have $(B_1+\cdots+B_N)A(N-1)\to 0$, and therefore (2.17) becomes

$$\begin{aligned}|S(z)| &\leq 4\sum_{m=1}^{\infty}(B_1+\cdots+B_m)(A(m-1)-A(m)) \\ &\leq 4C\sum_{m=1}^{\infty} m(A(m-1)-A(m))\end{aligned}$$

$$\leq C' \int_0^\infty A(t)\, dt.$$

Thus the proof will be completed after we prove (2.18) with suitable choices of C_k s.t. $C_k(\lambda_k) = 1$.

Now we prove (2.18), recall that by definition

$$B_1 + \cdots + B_m = \sum_{i=1}^{m} \sum_{k \in E_i} |C_k(z)|(1 - |a_{\lambda_k}(z)|^2)\delta_k^{p-1}. \tag{2.19}$$

We define $C_k(z)$ as follows. Let $A_k(z) = \prod_{i>k} a_{\lambda_i}(z)$, which is a nonvanishing analytic function since $|1 - a_{\lambda_n}(z)| \leq 2(1 - |\lambda_n|)/(1 - |z|)$. Moreover, we have

$$|A_k(\lambda_k)|^2 = \prod_{i>k} \frac{1}{|\lambda_i|} \left| \frac{\lambda_i - |\lambda_i|\lambda_k}{1 - \overline{\lambda_i}\lambda_k} \right|^2 \geq \prod_{i>k} \left| \frac{\lambda_i - \lambda_k}{1 - \overline{\lambda_i}\lambda_k} \right|^2 \geq \delta_k^2$$

for every k. Now we let

$$C_k(z) = \frac{[A_k(z)]^{p-1}}{[A_k(\lambda_k)]^{p-1}}$$

by fixing a branch of log. It then follows that

$$\begin{aligned} |C_k(z)|(1 - |a_{\lambda_k}(z)|^2)\delta_k^{p-1} &\leq \frac{|A_k(z)|^{p-1}}{\delta_k^{p-1}} 2(1 - |a_{\lambda_k}(z)|)\delta_k^{p-1} \\ &\leq 2|A_k(z)|^{p-1}(1 - |a_{\lambda_k}(z)|^{p-1}) \qquad (2.20) \\ &= 2(|A_k(z)|^{p-1} - |A_{k-1}(z)|^{p-1}). \end{aligned}$$

We combine this estimate with (2.19) to obtain

$$B_1 + \cdots + B_m \leq 2 \sum_{i=1}^{m} \underbrace{\sum_{k=1}^{\infty} (|A_k(z)|^{p-1} - |A_{k-1}(z)|^{p-1})}_{= \lim_{N\to\infty}(|A_N(z)|^{p-1} - |A_0(z)|^{p-1}) \leq 1} \leq 2m,$$

as desired.

To sum up, our interpolation is given by

$$\boxed{\sum_{n=1}^{\infty} a_n \left(\frac{1 - |\lambda_n|^2}{1 - \overline{\lambda_n} z} \right)^2 \frac{B_n(z)}{B_n(\lambda_n)} \cdot \frac{[A_n(z)]^{p-1}}{[A_n(\lambda_n)]^{p-1}}.}$$

∎

Chapter 3

Extremal Problems Through Duality

3.1 Preparative Results

Before going through the next theorem, we list out a few more facts that will be found important to us later.

Lemma 3.1. *Let $f \in H^1$. The Fourier coefficient of the nontangential limit $f(\theta)$ of $f(z)$ coincides with the Laurent series coefficient of $f(z)$ at* 0.

Proof. This follows from explicit computation. For every $r \in (0,1)$, let $n \in \mathbb{Z}$, then the n-th Laurent series coefficient is

$$\frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}}\,dz = \frac{1}{r^n}\int_0^{2\pi} f_r(e^{i\theta})e^{-in\theta}\,\frac{d\theta}{2\pi}.$$

On the other hand, since $\hat{f}(n) := \int_0^{2\pi} f(\theta)e^{-in\theta}\,\frac{d\theta}{2\pi}$, a triangle inequality gives

$$\left|\frac{f^{(n)}(0)}{n!} - \hat{f}(n)\right| \le (r^{-n}-1)\|f_r\|_1 + \|f_r - f\|_1$$

for every $r \in (0,1)$, when $r \to 1^-$, (b) of Theorem 1.5 gives $\hat{f}(n) = \frac{f^{(n)}(0)}{n!}$. ∎

As a consequence, when $f \in H^1$ ($\supseteq H^p$ for all $p \ge 1$), the negative Fourier coefficients vanish, i.e., $\hat{f}(n) = 0$ for all $n < 0$. Conversely, H^p is the set class of functions in $L^p(T)$ (where $p \geq 1$) such that $\hat{f}(n) = 0$ for all $n < 0$. This is simply because

$$P_r(t) = 1 + \sum_{n=1}^{\infty} r^n (e^{int} + e^{-int}).$$

Thus $P[f] = f \in H^p$ by Theorem 1.7 when identified as an element in $L^p(T)$.

Lemma 3.2. *Let $p \geq 1$, if g_n's $\in H^p$ are bounded in H^p and converge to g normally, then $g \in H^p$ and for fixed $k \in \mathbb{Z}$ the k-th Fourier coefficient of g_n converges to that of g. In other words, for any $k \in \mathbb{Z}$,*

$$\lim_{n\to\infty} \int_0^{2\pi} g_n e^{-ikt} \frac{dt}{2\pi} = \int_0^{2\pi} g e^{-ikt} \frac{dt}{2\pi}.$$

Proof. $g \in H^p$ since $\|g_r\|_p < \infty$ as $r \to 1$. Now the convergence of the kth Taylor coefficient of g_n (in n) follows from normal convergence:

$$\int_0^{2\pi} r^{-n} g_n(re^{it}) e^{-ikt}\, dt/2\pi \to \int_0^{2\pi} r^{-n} g(re^{it}) e^{-ikt}\, dt/2\pi.$$

Since Taylor coefficients and Fourier coefficients coincide, we are done. ∎

Lemma 3.3. *Let $f \in H^1(\mathbb{D})$ and let $f(t)$ be real a.e. on T, then $f(z)$ can be extended analytically across any point of T.*

The idea of the proof will borrow from that of the standard fact that if a continuous $f : D \to \mathbb{C}$ is analytic on $D \setminus \Gamma$ with D a domain and Γ a piecewise smooth Jordan arc that chops D into two pieces, then $f(z)$ is analytic on D. Thanks to the existence of nontangential limits, we have a natural candidate of extension across T:

Proof. Set $f(z) = \overline{f(1/\overline{z})}$ when $|z| > 1$. Let $L_i(r)$ and $T_i(r)$, $i = 1, 2, 4$, be defined roughly as in Figure 3.1, where L_3 and T_3 are fixed arc and independent of r, $L_1(r), T_1(r)$ are arcs with radius r and $2 - r$ respectively.

We denote

$$L(r) = (L_1 + L_2 + L_3 + L_4)(r) \quad \text{and} \quad T(r) = (T_1 + T_2 + T_3 + T_4)(r)$$

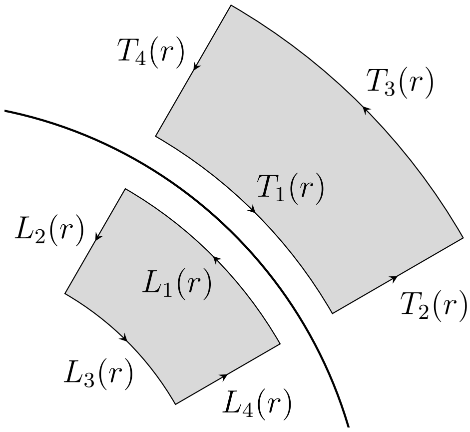

Figure 3.1: The contours we use.

the boundaries of corresponding annular sectors. Now for every $z \in \mathbb{D}$, for big enough r we have

$$f(z) = \left(\int_{L(r)} + \int_{T(r)} \right) \frac{f(\zeta)}{\zeta - z} \frac{d\zeta}{2\pi i}.$$

Of course the second integral vanishes, thus the integral formula follows, now we expand all contours:

$$f(z) = \left(\left[\int_{(L_2+L_3+L_4)(r)} + \int_{(T_2+T_3+T_4)(r)} \right] + \left[\int_{L_1(r)} - \int_{-T_1(r)} \right] \right) \frac{f(\zeta)}{\zeta - z} \frac{d\zeta}{2\pi i}.$$

The second square bracket converges to 0 by part (b) of Theorem 1.5 and by the fact that $f(t)$ is real a.e. on T, it follows that as $r \to 1^-$,

$$f(z) = \int_C \frac{f(\zeta)}{\zeta - z} \frac{d\zeta}{2\pi i},$$

where $C = (L_2 + L_3 + L_4 + T_2 + T_3 + T_4)(1)$ is a piecewise C^1 closed path. ∎

Lemma 3.4. *Let $D \subseteq \mathbb{C}$ be a domain such that $D \supseteq \overline{\mathbb{D}}$. Suppose $f : D \to \mathbb{C}$ is analytic and $f|_T \geq 0$, then every zero of $f(z)$ on T must have even multiplicity.*

Proof. Suppose that $z_0 \in T$ is a zero of $f(z)$, w.l.o.g. let's take $z_0 = 1$, then $f(z) = (z-1)^n g(z)$ for some $g(z)$ analytic near 1 with $g(1) \neq 0$. For every $z \in T \setminus \{1\}$ close enough to 1, we have

$$n \operatorname{Arg}(z-1) + \operatorname{Arg} g(z) \equiv 0 \pmod{2\pi},$$

where Arg denotes any branch of argument whose branch cut does not intersect $g(1)$, so as to make $\operatorname{Arg} g(z)$ continuous near $z = 1$.

Let's take $z_1 \in T \setminus \{1\}$ near 1 and then take $z_2 \in T$ such that $z_1 - 1 = \overline{z_2 - 1}$, we have

$$n \operatorname{Arg}(z_1 - 1) + \operatorname{Arg} g(z_1) \equiv 0 \pmod{2\pi}$$

and

$$-n \operatorname{Arg}(z_1 - 1) + \operatorname{Arg} g(z_2) \equiv 0 \pmod{2\pi},$$

hence

$$2n \operatorname{Arg}(z_1 - 1) + \operatorname{Arg} g(z_1) - \operatorname{Arg} g(z_2) \equiv 0 \pmod{2\pi}.$$

If we take $T \ni z_1 \to 1$, then the equality above modulo 2π becomes

$$2n \left(\frac{\pi}{2}\right) + 0 \equiv 0 \pmod{2\pi},$$

which is the same as saying n is even. ∎

3.2 Extremal Problems

This section is devoted to solving extremal problems in Hardy spaces. In handling extremal problems we will apply the following well-known theorem to switch a problem from finding maximum to finding minimum:

Theorem 3.5 (Duality). *Let M be a closed vector subsapce of a normed space X. We have the following isometric isomorphisms and equations.*

(a) $M^* = X^*/M^\perp$. *For every $F \in X^*$,*

$$\sup\{|\langle x, F\rangle| : x \in M, \|x\| \le 1\} = \min\{\|F - G\| : G \in M^\perp\}.$$

(b) $(X/M)^* = M^\perp$. *For every $x \in X$,*

$$\inf\{\|x - m\| : m \in M\} = \max\{|\langle x, G\rangle| : G \in M^\perp, \|G\| \le 1\}.$$

3.2.1 In Hardy Spaces

Our first two extremal results will be done in H^2 since the Parseval's identity in Fourier series makes the explicit computation possible.

We now improve one of the consequences of Corollary 1.12 in the sense that the constant in the definition of boundedness of $i_z : H^p \to \mathbb{C}$ is optimally sharpened:

Theorem 3.6. *If $f \in H^p$ $(0 < p \leq \infty)$, then*

$$|f(z)| \leq (1 - |z|^2)^{-1/p} \|f\|_p.$$

Moreover, the inequality is sharp for each fixed z.

Note by sharpness we mean for each $z \in \mathbb{D}$ a solution $f \in H^p$ such that $\|f\|_p = 1$ with $|f(z)| = (1-|z|^2)^{-1/p}$. Thus f is an "extremal function" attaining

$$\max\{|i_z(f)| : f \in H^p, \|f\|_p = 1\}.$$

Note that the maximum exists simply by a normal family argument *due to boundedness w.r.t. H^p-norm*, which will be the technique that we always use in our later theorem in showing existence of "extremal function".

Proof. The case $p = \infty$ is trivial. Suppose the case $p = 2$ was done at the moment. Let B be the Blaschke product of zeros of f, then there is an $h \in H^2$ such that $f = Bh^{2/p}$ with $\|h\|_2^2 = \|f\|_p^p$, then

$$|h(z)| \leq (1 - |z|^2)^{-1/2} \|h\|_2 = (1 - |z|^2)^{-1/2} \|f\|_p^{p/2},$$

this implies

$$|h(z)|^{2/p} \leq (1 - |z|^2)^{-1/p} \|f\|_p.$$

Since Blaschke product is inner, $|B| \leq 1$, thus

$$|f(z)| = |B(z)||h(z)|^{2/p} \leq |h(z)|^{2/p} \leq (1 - |z|^2)^{-1/p} \|f\|_p.$$

Therefore it remains to prove the case $p = 2$.

Suppose now $p = 2$, let $i_z \in (H^2)^*$ be the pointwise evaluation at z, it is enough to show

$$\|i_z\| = (1 - |z|^2)^{-1/2}.$$

For this, note that

$$\|i_z\| = \sup_{\substack{f\in H^2\\ \|f\|_2\le 1}} |f(z)| = \sup_{\substack{f\in H^2\\ \|f\|_2\le 1}} \int_0^{2\pi} P_z(t)f(t)\,\frac{dt}{2\pi} = \sup_{\substack{f\in H^2\\ \|f\|_2\le 1}} P_z(f). \tag{3.7}$$

Part (a) of Theorem 3.5 yields

$$\sup_{\substack{f\in H^2\\ \|f\|_2\le 1}} P_z(f) = \min_{g\in H_0^2} \|P_z - g\|_2.$$

Now we can minimize RHS by using Fourier series method. Recall that if $z = re^{i\theta}$, then $P_z(t) = P_r(\theta - t)$, it follows that

$$\begin{aligned}\min_{g\in H_0^2} \|P_z - g\|_2 &= \min_{g\in H_0^2} \|P_r(t) - g(t)\|_2\\ &= \min_{g\in H_0^2} \left(\sum_{n=-\infty}^{0} |\widehat{P_r}(n)|^2 + \sum_{n=1}^{\infty} |\widehat{P_r}(n) - \widehat{g}(n)|^2\right)^{1/2}.\end{aligned}$$

Recall that $P_r(t) = \sum_{n\in\mathbb{Z}} r^{|n|}e^{int}$, we may take $g_0(z) = \sum_{n=1}^{\infty} r^n z^n \in H_0^2$ to get desired minimization, therefore

$$\min_{g\in H_0^2} \|P_z - g\|_2 = \left(\sum_{n=-\infty}^{0} (r^{|n|})^2\right)^{1/2} = \frac{1}{(1-r^2)^{1/2}} = \frac{1}{(1-|z|^2)^{1/2}}.$$

For sharpness, a normal family argument shows that there is an $f_0 \in H^2$ with $\|f_0\|_2 = 1$ such that

$$|f_0(z)| = |i_z(f_0)| = \|i_z\| = (1-|z|^2)^{-1/2}\|f_0\|_2.$$

Let B be the Blaschke product of the zeros of f_0, set $u = f_0/B$, then u is nonvanishing and

$$|f_0(z)| \le |u(z)| \le (1-|z|^2)^{-1/2}\|u\|_2 = (1-|z|^2)^{-1/2}\|f_0\|_2.$$

This shows that $|u(z)| = (1-|z|^2)^{-1/2}\|u\|_2$. Now we simply take $v = u^{2/p}$, then $|v(z)| = (1-|z|^2)^{-1/p}\|v\|_p$, thus the inequality is sharp, i.e.,

$$\|i_z\|_{(H^p)^*} = (1-|z|^2)^{-1/p}. \qquad \blacksquare$$

Theorem 3.8. *For fixed z, $|z| < 1$, and for each positive integer n, the maximum of $|f^{(n)}(z)|$ over all H^2 function f with $\|f\|_2 \le 1$ is*

$$n!\left(\sum_{k=0}^{\infty}\binom{n+k}{n}^2|z|^{2k}\right)^{1/2}.$$

The solution to its dual extremal problem in (3.9) is

$$g_0(w) = \sum_{k=1}^{\infty}(-1)^n\binom{n+k-1}{n}w^k.$$

Proof. By Cauchy integral formula we have

$$\sup_{\substack{f\in H^2\\ \|f\|_2\le 1}} |f^{(n)}(z)| = n! \sup_{\substack{f\in H^2\\ \|f\|_2\le 1}} \left|\int_0^{2\pi} \frac{e^{i\theta}}{(e^{i\theta}-z)^{n+1}} f(\theta)\,\frac{d\theta}{2\pi}\right|.$$

We call finding the supremum an (original) extremal problem.

Define

$$F_n(\theta) = \frac{e^{i\theta}}{(e^{i\theta}-z)^{n+1}},$$

then by part (a) of Theorem 3.5,

$$M := \sup_{\substack{f\in H^2\\ \|f\|_2\le 1}} |F_n(f)| = \min_{g\in H_0^2} \|F_n(\theta) - g\|_2. \tag{3.9}$$

We call finding the minimum above a dual extremal problem. The extremal function of the original problem (the leftmost quantity in (3.9)) exists by a normal family argument. The dual extremal function exists automatically by the Duality Theorem just used.

For $f_0 \in H^2$ and $g_0 \in H_0^2$ to be the corresponding extremal functions, they are necessary and sufficient to satisfy

$$\left|\int_0^{2\pi} F_n f_0 \,\frac{d\theta}{2\pi}\right| = \|F_n(\theta) - g_0\|_2. \tag{3.10}$$

this is easily seen from the equality (3.9). Thus, generally we have two approaches in finding M. Either we raise an example such that (3.10) holds, or we directly maximize the LHS $\big/$ minimize the RHS, if possible. The latter approach is much more feasible in this proof.

Let

$$g_0(w) = \sum_{n=1}^{\infty} b_n w^n \in H_0^2,$$

from Fourier series we have

$$\text{RHS of (3.10)} = \left(\sum_{k=-\infty}^{\infty} |\widehat{(F_n - g_0)}(k)|^2 \right)^{1/2}.$$

Note that since $g_0 \in H_0^2$, by Lemma 3.1 we have $\widehat{g_0}(k) = 0$ when $k \le 0$ and $\widehat{g_0}(k) = b_k$ when $k \ge 1$. Moreover,

$$\begin{aligned}
\widehat{F_n}(k) &= \int_0^{2\pi} \frac{e^{i(1-k)\theta}}{(e^{i\theta} - z)^{n+1}} \frac{d\theta}{2\pi} \\
&= \frac{1}{n!} \frac{n!}{2\pi i} \int_T \frac{\zeta^{-k}}{(\zeta - z)^{n+1}} \, d\zeta \\
&= \frac{1}{n!} (-k)(-k-1) \cdots (-k-(n-1)) \frac{1}{z^{n+k}}.
\end{aligned}$$

From this we can conclude

$$\begin{aligned}
\text{RHS of (3.10)} &= \left(\begin{array}{l} \displaystyle\sum_{k \le 0} \left(\frac{1}{n!} (-k)(-k-1) \cdots (-k-(n-1)) \frac{1}{|z|^{n+k}} \right)^2 \\ \displaystyle\qquad + \sum_{k=1}^{\infty} \left| \frac{(-1)^n}{n!} \frac{(n+k-1)!}{(k-1)!} \frac{1}{z^{n+k}} - b_k \right|^2 \end{array} \right)^{1/2} \\
&= \left(\sum_{k=0}^{\infty} \binom{n+k}{n}^2 |z|^{2k} + \sum_{k=1}^{\infty} \left| (-1)^n \binom{n+k-1}{n} - b_k \right|^2 \right)^{1/2}.
\end{aligned}$$

Therefore we minimize RHS above to get

$$M = \left(\sum_{k=0}^{\infty} \binom{n+k}{n}^2 |z|^{2k} \right)^{1/2},$$

and the dual extremal function is

$$g_0(w) = \sum_{k=1}^{\infty} (-1)^n \binom{n+k-1}{n} w^k.$$

We conclude that

$$\max_{\substack{f \in H^2 \\ \|f\|_2 \le 1}} |f^{(n)}(z)| = n! M = n! \left(\sum_{k=0}^{\infty} \binom{n+k}{n}^2 |z|^{2k} \right)^{1/2}. \qquad \blacksquare$$

The original extremal function may be hard to find. But that's natural since there is no guarantee on the feasibility of finding explicit solution.

3.2.2 A Coefficient Problem of Bounded Analytic Functions

Given $c_0, c_1, \ldots, c_N \in \mathbb{C}$, we call finding the maximum of $|c_0a_0+c_1a_1+\cdots+c_Na_N|$ over $f = \sum a_j z_j \in H^p$ a coefficient problem.

The special case that $p = \infty$ and $c_0 = c_1 = \cdots = c_N = 1$ were posed and solved by E. Landau. We now consider $p = \infty$ with general $c_0, c_1, \ldots, c_N \in \mathbb{C}$.

Theorem 3.11. *Let $c_0, c_1, \ldots, c_N$ be given complex numbers and consider the maximum problem*

$$M = \sup_{\substack{f \in H^\infty \\ \|f\| \le 1}} \left| \sum_{j=0}^{N} c_j a_j \right|,$$

where $f(z) = \sum_{j=0}^{\infty} a_j z^j$. The dual extremal problem is

$$M = \inf_{g \in H_0^1} \|k - g\|_1,$$

where $k(z) = \sum_{j=0}^{n} c_j z^{-j}$. Moreover, it is equivalent to the minimum problem

$$M = \inf\{\|h\|_1 : h \in H^1, h = c_N + c_{N-1}z + \cdots + c_0 z^N + \cdots\}.$$

Theorem 3.11 was studied by F. Riesz in 1920 using variational method in [RIE]. Here we study the same problem through functional analysis on H^p.

Proof. Since $f(z) = \sum_{j \ge 0} a_j z^j$, by Lemma 3.1 we have

$$a_j = \int_0^{2\pi} f(t) e^{-ijt} \frac{dt}{2\pi},$$

therefore we have

$$\begin{aligned} M &= \sup_{\substack{f \in H^\infty \\ \|f\| \le 1}} \left| \sum_{j=0}^{N} c_j a_j \right| \\ &= \sup_{\substack{f \in H^\infty \\ \|f\| \le 1}} \left| \int_0^{2\pi} \left(\sum_{j=0}^{N} c_j (e^{it})^{-j} \right) f(t) \frac{dt}{2\pi} \right| \\ &= \sup_{\substack{f \in H^\infty \\ \|f\| \le 1}} \left| \int_0^{2\pi} k(e^{it}) f(t) \frac{dt}{2\pi} \right|, \qquad (3.12) \end{aligned}$$

where

$$k(z) = \sum_{j=0}^{N} c_j z^{-j}. \tag{3.13}$$

Thus M is the operator norm of k in $(H^\infty)^*$ when viewed as a linear functional on H^∞. Although we have the following duality relation:

$$(H^\infty)^* = (L^\infty)^*/(H^\infty)^\perp = (L^\infty)^*/H_0^1, \tag{3.14}$$

but $(L^\infty)^*$ cannot be viewed as a subspace of L^1. In fact, conversely, L^1 is a proper subspace of $(L^\infty)^*$. Nevertheless, it still suggests we should have

$$M = \min_{g \in H_0^1} \|k - g\|_1.$$

Fortunately this remains true, but this is not a direct consequence of Theorem 3.5, i.e., it does not directly follow from (3.14).

From p. 313 of [DUR]: for every $h \in L^1$, we have

$$\sup_{\substack{f \in H^\infty \\ \|f\| \le 1}} \left| \int_0^{2\pi} h(t) f(t) e^{it} \, \frac{dt}{2\pi} \right| = \min_{g \in H^1} \|h - g\|_1.$$

We take $h = e^{-it} k$ above to conclude that

$$\begin{aligned} M = \sup_{\substack{f \in H^\infty \\ \|f\| \le 1}} \left| \int_0^{2\pi} k(t) f(t) \, \frac{dt}{2\pi} \right| &= \min_{g \in H^1} \|e^{-it} k - g\|_1 \\ = \min_{g \in H^1} \|k - e^{it} g\|_1 = \min_{g \in H_0^1} \|k - g\|_1 &= d(k, H_0^1), \end{aligned} \tag{3.15}$$

as desired.

Since every $g \in H_0^1$ is of the form $g(z) = b_1 z + b_2 z^2 + \cdots$, therefore by $\|k - g\|_1 = \|z^N k - z^N g\|_1$ we have

$$M = \inf\{\|h\|_1 : h \in H^1, h = c_N + c_{N-1} z + \cdots + c_0 z^N + \text{higher order terms}\}.$$

Recall the definition of $k(z)$ in (3.13) if the reader get confused. ∎

Theorem 3.16. *In Theorem 3.11 the original extremal problem has a unique extremal function $f_0 \in H^\infty$ and the dual problem has a unique minimizing function (or dual extremal function) $g_0 \in H_0^1$. Moreover, with the same notation in Theorem 3.11,*

$$f_0(z)(k(z) - g_0(z)) = |k(z) - g_0(z)|, \quad |z| = 1,$$

so that

$$f_0(z)(k(z)-g_0(z)) = cz^q \prod_{j=1}^{n}(z-\alpha_j)(1-\overline{\alpha_j}z)\Big/ z^N$$

with $n+q=N$, $0<|\alpha_j|\le 1$, $c>0$. *Reindexing* $\alpha_1,\ldots,\alpha_n$ *if necessary, there is* $s\in\mathbb{N}$, $0\le s\le n$, *such that* $|\alpha_j|<1$, $j\le s$ *with*

$$f_0(z) = \gamma z^q \prod_{j=1}^{s}\left(\frac{z-\alpha_j}{1-\overline{\alpha_j}z}\right)$$

and

$$k(z)-g_0(z) = c\overline{\gamma}\prod_{j=1}^{n}(1-\overline{\alpha_j}z)^2 \prod_{j=s+1}^{n}\left(\frac{z-\alpha_j}{1-\overline{\alpha_j}z}\right)\Big/ z^N,$$

where $|\gamma|=1$.

Proof. The existence of solution $g_0\in H_0^1$ of the dual extremal problem is clear from the fact cited in (3.15). We establish the existence of extremal solution—$f_0\in H^\infty$ with $\|f_0\|_\infty\le 1$—of our original problem (3.12) now.

Existence. Let's prove the existence as follows. In view of (3.12), we let $f_n\in H^\infty$ and $\|f_n\|\le 1$ be such that

$$0\le \int_0^{2\pi} k(t)f_n(t)\frac{dt}{2\pi}\to M.$$

Since $\{f_n\}$ is uniformly bounded, it forms a normal family. Passing to further subsequence if necessary, we may assume f_n itself converges normally to a function $f_0\in H^\infty$, then by Lemma 3.2 we have

$$\int_0^{2\pi} kf_n\,\frac{dt}{2\pi}\to\int_0^{2\pi} kf_0\,\frac{dt}{2\pi}.$$

On the other hand, since $\int_0^{2\pi} kf_n\,dt/2\pi\to M$, we have $\int_0^{2\pi} kf_0\,dt/2\pi = M$. Since $\|f_n\|\le 1$ for each n, we have $\|f_0\|\le 1$, so $f_0\in H^\infty$. Thus (3.12) has a solution.

Uniqueness. Next we focus on the uniqueness of the solutions of our extremal problems. Let $f_0\in H^\infty$ and $g_0\in H_0^1$ be the extremal solutions, then we have

$$M=\int_0^{2\pi} f_0(k-g_0)\,\frac{dt}{2\pi}\le\int_0^{2\pi}|f_0||k-g_0|\,\frac{dt}{2\pi}\le\|f_0\|_\infty\|k-g_0\|_1\le 1\cdot\|k-g_0\|_1 = M.$$

Thus the equality of each inequality must be attained thoroughly. Therefore

$$f_0(k-g_0) = |f_0||k-g_0| \quad \text{and} \quad \|f_0\|_\infty = 1.$$

If $k-g_0$ is zero on a set of positive measure of T, then so is $z^N(k-g_0)$, therefore by Corollary 1.12, $z^N(k-g_0) \equiv 0$ on $\mathbb{D}$, thus $k \equiv g_0$ on $\mathbb{D}$, a contradiction. So $k-g_0$ is nonzero almost everywhere, and hence by $\int_0^{2\pi} |f_0||k-g_0| \frac{dt}{2\pi} = \|k-g_0\|$ we have $|f_0||k-g_0| = |k-g_0|$ a.e. and thus $|f_0| = 1$ a.e.. It follows that

$$f_0 = \frac{|k-g_0|}{k-g_0} \text{ a.e..} \tag{3.17}$$

This shows that f_0 is uniquely determined.

Now $\mathrm{Re}[if_0(k-g_0)] = 0$ on T, thus we have $\mathrm{Re}(if_0g_0) = \mathrm{Re}(if_0k)$. Since $if_0g_0 \in H_0^1$, the real part of if_0g_0 uniquely determine if_0g_0. As f_0 is uniquely determined, so is g_0. So the uniqueness of the pair f_0, g_0 is established.

Expression. Since $f(k-g_0)$ is real a.e. on T, it can be extended analytically across T by Lemma 3.3 applied to $F = f(k-g_0)$. By setting $F(z) = \overline{F(1/\overline{z})}$ on $|z| > 1$ the function $f_0(k-g_0)$ extends to a meromorphic function on $\widehat{\mathbb{C}}$, with two poles 0 and ∞ that are not essential singularity, hence rational.

Due to the way of extension, if α is a zero of $f_0(k-g_0)$ in $|z| < 1$, then $1/\overline{\alpha}$ is a zero of $f_0(k-g_0)$ in $|z| > 1$, we conclude

$$f_0(k-g_0)(z) = c\frac{z^q \prod_{j=1}^n (z-\alpha_j)(1-\overline{\alpha_j}z)}{z^N} \tag{3.18}$$

for some $0 < |\alpha_j| \leq 1$ and constant $c \in \mathbb{C}$. We allow $|\alpha_j| = 1$ in the expression because each root on the boundary must be of even multiplicity by Lemma 3.4 since $f_0(k-g_0) \geq 0$ a.e. on T.

By the way of extension the order of zero at 0 must be the same as that at infinity. As

$$f_0(k-g_0)(1/z) = c\frac{z^q \prod_{j=1}^n (1-\alpha_i z)(z-\overline{\alpha_i})}{z^{q+2n}},$$

we conclude

$$N - q - 2n = q - N \iff q + n = N.$$

Now we try to find the expression of f_0. In view of canonical factorization (see Theorem 1.21) we first try to clear out the Blaschke factor of f_0 and $k - g_0$. Reindexing if necessary, we let $\alpha_1, \dots, \alpha_s$ be the zeros of f_0 in $\mathbb{D}$, then

$$F = f_0 \Bigg/ z^q \left(\prod_{j=1}^{s} \frac{z - \alpha_j}{1 - \overline{\alpha_j} z} \right) \in H^\infty$$

and

$$G = (k - g_0) z^N \Bigg/ \prod_{j=s+1}^{n} \frac{z - \alpha_j}{1 - \overline{\alpha_j} z} \in H^1$$

are nonvanishing analytic functions on $\mathbb{D}$, i.e., they are just composed of inner and outer factor. Let's try to show F, G have no inner factor, to do this, it is enough to show $F \cdot G$ has no inner factor, i.e., it is outer.

We note that $F \cdot G$ is a rational function without zeros in $\mathbb{D}$, therefore $F \cdot G$ is a product of linear factors with zero on $|z| \geq 1$ (up to a multiplicative constant), it is enough to show $z - \alpha$ is outer when $|\alpha| \geq 1$. For this, it is enough to show $1/(z - \alpha) \in H^r$ for some $r > 0$, this is obvious when $|\alpha| > 1$; for $|\alpha| = 1$, this is true when $r = p$ for any $p \in (0, 1)$. To see this, write

$$\int_0^{2\pi} \frac{1}{|e^{i\theta} - \alpha|^p} \, d\theta = \left(\int_{|\theta| < \delta} + \int_{|\theta| \geq \delta} \right) \frac{1}{|1 - e^{i\theta}|^p} \, d\theta,$$

as $\lim_{\theta \to 0} \frac{|1 - e^{i\theta}|}{\theta} \to 1$, so the first integral is bounded for sufficiently small δ. For this fixed choice of δ, the second integral is also bounded, so $1/(z - \alpha) \in H^p$ for any $p \in (0, 1)$. Therefore $F \cdot G$ is a product of outer factors, which must also be outer, this establishes the fact that F and G are outer functions.

Since $F \in H^\infty$ is an outer function, it is uniquely determined by its magnitude on T, namely, $\ln |F|$ is integrable and

$$F = \gamma \exp \left(\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \ln |F(e^{it})| \frac{dt}{2\pi} \right).$$

Since $|F| = |f_0| = 1$ a.e. on T by the argument before (3.17), we have $F = \gamma$, therefore

$$f_0 = \gamma z^q \left(\prod_{j=1}^{s} \frac{z - \alpha_j}{1 - \overline{\alpha_j} z} \right),$$

where $|\gamma| = 1$. Plugging in this into the formula (3.18) we have

$$k - g_0 = c\overline{\gamma} \frac{z^q \prod_{j=1}^{n} (z - \alpha_j)(1 - \overline{\alpha_j} z)}{z^q \prod_{j=1}^{s} \frac{z - \alpha_j}{1 - \overline{\alpha_j} z}} \Bigg/ z^N$$

$$= c\overline{\gamma}\prod_{j=1}^{n}(1-\overline{\alpha_j}z)^2\prod_{j=s+1}^{n}\frac{z-\alpha_j}{1-\overline{\alpha_j}z}\Big/ z^N. \qquad \blacksquare$$

Theorem 3.19. *In Theorem 3.11, if $c_N \neq 0$, then there are $\lambda_j \in \mathbb{C}$ such that*

$$\left(\sum_{j=0}^{\infty}\lambda_j z^j\right)^2 = c_N + c_{N-1}z + \cdots + c_0 z^N + b_{N+1}z^{N+1} + b_{N+2}z^{N+2} + \cdots .$$

We set $P_N(z) = \sum_{j=0}^{N}\lambda_j z^j$. If $P_N(z)$ has no zero in $|z| \leq 1$, then there is an $n \leq N$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{D}$ such that

$$f_0(z) = \frac{\overline{c_N}}{|c_N|}z^{N-n}\prod_{j=1}^{n}\left(\frac{z-\alpha_j}{1-\overline{\alpha_j}z}\right)$$

is the extremal function. Moreover, the dual extremal function satisfies $k(z) - g_0(z) = P_N^2(z)/z^N$ and we have

$$M = \|P_N^2\|_1 = \sum_{j=0}^{N}|\lambda_j|^2.$$

Before going into the proof, observe that formally $(z^N k(z))^{1/2} = P_N(z) + z^N g(z)$, where $g(0) = 0$, and therefore

$$z^N k = P_N^2 + z^N g_0(z).$$

Now we still have $g_0(0) = 0$ and we have $k = P_N^2/z^N + g_0$, so

$$P_N^2/z^N = k - g_0$$

for some $g_0 \in H_0^1$. Hopefully and indeed g_0 is an extremal function. A similar construction can be done with $1/2$ replaced by $1/3, 1/4$, ..., whatever, but we prefer to have explicit computation by Fourier series method.

Proof. Now we observe that f_0, g_0 are the corresponding extremal functions if and only if

$$\left|\int_0^{2\pi} f_0(k-g_0)\,\frac{d\theta}{2\pi}\right| = \|k-g_0\|_1. \tag{3.20}$$

To be convinced the readers can look at inequalities (3.15). Now we aim at finding such $f_0 \in H^\infty$ and $g_0 \in H_0^1$.

Let $P_N(z) = \sum_{j=0}^{N} \lambda_j z^j$. Since $P_N(z)$ has no zero in $\overline{\mathbb{D}}$, there is an $n \leq N$ such that

$$P_N(z) = C_N \prod_{j=1}^{n} (1 - \overline{\alpha_j} z),$$

for some $C_N \in \mathbb{C}$. It follows that

$$z^{-N} P_N^2(z) = c_N z^{-N} \prod_{j=1}^{n} (1 - \overline{\alpha_j} z)^2.$$

This function is of the form $k - g_0$ for some $g_0 \in H_0^1$ mentioned before the proof. Construct

$$f_0(z) = \frac{c_N}{|c_N|} z^{N-n} \prod_{j=1}^{n} \left(\frac{z - \alpha_j}{1 - \overline{\alpha_j} z} \right),$$

then we find that

$$f_0(z) P_N^2(z)/z^N \geq 0 \quad \text{when } |z| = 1,$$

and hence

$$\int_0^{2\pi} \big(f_0(z) z^{-N} P_N^2(z)\big)(e^{i\theta}) \, \frac{d\theta}{2\pi} = \int_0^{2\pi} |z^{-N} P_N^2(z)|(e^{i\theta}) \, \frac{d\theta}{2\pi}.$$

Therefore (3.20) is satisfied, f_0 and $g_0 = k - p_N^2/z^N$ are the corresponding extremal functions.

Finally we have

$$M = \|z^{-N} P_N^2\|_1 = \|P_N^2\|_1 = \|P_N\|_2^2 = \sum_{j=0}^{N} |\lambda_j|^2. \qquad \blacksquare$$

Chapter 4

Future Research Direction

4.1 The Nevanlinna-Pick Problem

In 1916 G. Pick proved the following theorem:

Theorem 4.1 (Pick). *Let $z_1, \ldots, z_n \in \mathbb{D}$ and $w_1, \ldots, w_n \in \overline{\mathbb{D}}$. There exists a bounded analytic $f : \mathbb{D} \to \overline{\mathbb{D}}$ satisfying the interpolating*

$$f(z_j) = w_j, \quad j = 1, 2, \ldots, n \tag{4.2}$$

if and only if the quadratic form

$$Q_n(t_1, \ldots, t_n) := \sum_{j,k=1}^{n} \frac{1 - w_j \overline{w_k}}{1 - z_j \overline{z_k}} t_j \overline{t_k}$$

is nonnegative: $Q_n \geq 0$. When $Q_n \geq 0$, there is a Blaschke product of degree at most n which solves (4.2).

Nowadays the matrix

$$\boxed{M := \left(\frac{1 - w_j \overline{w_k}}{1 - z_j \overline{z_k}} \right)_{1 \leq i,j \leq n}} \tag{4.3}$$

is called the Nevanlinna-Pick matrix for $H^\infty(\mathbb{D})$, where $z_1, \ldots, z_n \in \mathbb{D}$ and $w_1, \ldots, w_n \in \overline{D}$. Finding a function in a corresponding function space that satisfies (4.2) is called Nevallina-Pick problem. For example, it is known that:

Theorem 4.4. *Let H be a reproducing kernel space on $\mathbb{D}$. Let $z_1, z_2, \ldots, z_n \in \mathbb{D}$ be distinct and $w_1, w_2, \ldots, w_n \in \mathbb{C}$. Suppose that the reproducing kernels $K_1, K_2, \ldots, K_n$, at $z_1, \ldots, z_n$ respectively, are linearly indepednent, then the following are equivalent:*

(i) There exists an $f \in H$ such that $\|f\| \le 1$ and $f(z_i) = w_i$ for $i = 1, 2, \ldots, n$;

(ii) The Nevanlinna-Pick matrix $M = (\langle K_i, K_j\rangle - \overline{w_i}w_j)_{1\le i,j\le n}$ is nonnegative.

Furthermore, it is proved in [LI] that the above are also equivalent to

(iii) $\det M \ge 0$, where M is the matrix in (ii).

Consider $H = H^2(\mathbb{D})$, then the reproducing kernels at z_i is given by $K_i(z) = 1/(1 - \overline{z_i}z)$, and the Nevanlinna-Pick matrix for $H^2(\mathbb{D})$ is then

$$\boxed{M = \left(\frac{1 - (1 - \overline{z_i}z_j)\overline{w_i}w_j}{1 - \overline{z_i}z_j}\right)_{1\le i,j\le n}.}$$

Comparing the Nevanlinna-Pick matrix for H^∞ in (4.3) and that for H^2 above, it is conjectured:

Conjecture 4.5. *The solvability of the Nevanlinna-Pick problem on $H^p(\mathbb{D})$ (with some $f(z)$ s.t. $\|f\|_p \le 1$, $0 < p \le \infty$) is equivalent to*

$$\boxed{M_p = \left(\frac{1 - (1 - \overline{z_i}z_j)^{2/p}\overline{w_i}w_j}{1 - \overline{z_i}z_j}\right)_{1\le i,j\le n} \ge 0,}$$

where the principal branch of logarithm is taken in each entry.

Objective. Show that, at least, the nonnegativity M_p is a necessary condition for solvability of the Nevanlinna-Pick problem in H^p.

Give give two examples as evidence. When $n = 1$, $z_1 = z$ and $w_1 = f(z)$, then $M_p \ge 0$ is the same as

$$(1 - |z|^2)^{-1/p} \ge |f(z)|,$$

which is true by Theorem 3.6 with $\|f\|_p \leq 1$.

Also, set $n = 2$, and it is no loss of generality to assume further that $z_1 = 0$, then set $z_2 = z$, $w_1 = f(0)$ and $w_2 = f(z)$, the condition $M_p \geq 0$ becomes

$$[1 - |f(0)|^2][1 - (1 - |z|^2)^{2/p}|f(z)|^2] \geq [1 - |z|^2]\left|1 - \overline{f(0)}f(z)\right|^2. \tag{4.6}$$

Suppose it happens that $f(0) = 0$, then (4.6) becomes

$$\begin{aligned} & 1 - (1 - |z|^2)^{2/p}|f(z)|^2 \geq 1 - |z|^2 \\ \iff \quad & |z| \geq (1 - |z|^2)^{1/p}|f(z)| \\ \iff \quad & (1 - |z|^2)^{-1/p} \geq |g(z)|, \end{aligned}$$

where $f(z) = zg(z)$, which again holds by Theorem 3.6 since $\|g\|_p = \|f\|_p \leq 1$ in H^p.

It remains in progress to determine whether or not the conjecture holds when $n \geq 2$.

4.2 Interpolating Blaschke Products

We start off by introducing the standard terminology in potential theory on the complex plane. We will then describe a standard fact on approximation of inner functions by Blaschke product in Corollary 4.12, which will lead us to a natural open problem.

Definition 4.7. For a compactly supported Borel measure μ on $\mathbb{C}$ we define the *energy* of μ by

$$I(\mu) = \int_{\mathbb{C}}\int_{\mathbb{C}} \ln|x - y|\, d\mu(x)d\mu(y).$$

It it obvious that $I(\mu) < \infty$ for any compactly supported positive measure on $\mathbb{C}$, therefore μ has finite energy if and only if $I(\mu) > -\infty$.

Definition 4.8. A set E is said to be *polar* if for every positive Borel measure μ compactly supported in E, we have $I(\mu) > -\infty \implies \mu = 0$.

In other words, any nonzero measure supported on a polar set must have unbounded energy. For example, any singleton in $\mathbb{C}$ must be polar. A standard result in potential theory says that a countable union of polar sets is polar, therefore all countable sets are polar.

Definition 4.9. For any set E, we define the *capacity of E* by

$$c(E) := \exp\left(\sup_{\operatorname{spt}(\mu) \Subset E, \mu(\mathbb{C})=1} I(\mu) \right).$$

As easily seen, since $e^{-\infty} = 0$, so that $c(E) = 0$ precisely when a set E is polar. Since our main purpose up to this point is to develop a vocabulary to describe Corollary 4.12, we will not go any further to capacity theory.

Theorem 4.10. *Let μ be a finite Borel measure on $\mathbb{C}$ with compact support, and suppose that $I(\mu) > -\infty$, then $\mu(E) = 0$ for every Borel polar set.*

For a proof, see p. 56 of [RAN]. By Theorem 4.10 we have $c(E) = 0 \implies A(E) = 0$, where A is the area measure on $\mathbb{C}$, this follows from the fact that $A \llcorner B(0, \rho)$ has finite energy for any $\rho > 0$. Therefore the complement of a polar set in $\mathbb{C}$ must be dense in $\mathbb{C}$.

Theorem 4.11 (Frostman). *Let $f(z)$ be a nonconstant inner function on the unit disk $\mathbb{D}$. Then for nearly every $\zeta \in \partial\mathbb{D}$, i.e., excpet possibly for a set of capacity zero, the function*

$$f_\zeta(z) = \frac{f(z) - \zeta}{1 - \bar{\zeta} f(z)}$$

is a Blaschke product.

For a proof, see p. 76 of [GAR].

Corollary 4.12. *The set of Blaschke products is uniformly dense in the set of inner functions.*

Proof. By Theorem 4.11 and a direct consequence of Theorem 4.10, we deduce that for A-a.e. $\zeta \in \mathbb{D}$, f_ζ is a Blaschke product, and since

$$\|f - f_\zeta\|_\infty \leq \frac{2|\zeta|}{1 - |\zeta|},$$

so given $\epsilon > 0$, $|\zeta|$ can be chosen small with $\|f - f_\zeta\|_\infty < \epsilon$ and f_ζ being a Blaschke product. ∎

A relatively longer proof of Theorem 4.11 can be found in pp. 83-86 of [KOO] which avoids any use of tools from potential theory.

Now the following open problem becomes natural to us:

Open Problem. Can every Blaschke product be uniformly approximated by interpolating Blaschke products?

Here an interpolating Blaschke product means it is the Blaschke product of an interpolating sequence.

When the zeros of a Blaschke product are distributed nice enough, a positive answer was found by Li in [LI2]:

Theorem 4.13. *If a Blaschke product has all its zeros lying on finitely many radii, then it can be uniformly approximated by interpolating Blaschke products.*

Objective. Improve Li's result.

Bibliography

[DUR] P. L. Duren, *Theory of H^p Spaces*, Academic Press, 1970.

[GAR] John B. Garnett, *Bounded Analytic Functions*, revised 1st ed., Springer-Verlag, New York, 2007.

[GAR2] John B. Garnett, *Two Remarks on Interpolation By Bounded Analytic Functions*, Springer, LNM 604, pp. 32-40, 1977.

[JON] P. Jones, *Free Interpolation in H^∞*, J. Soviet Math. 22, pp. 1838-1839, 1983.

[KOO] Paul Koosis and V.P. Havin, *Introduction to H_p Spaces*, 2nd ed., Cambridge University Press, 1998.

[LI] Kin Y. Li, Nevanlinna-Pick Problem on Reproducting Kernel Spaces, preprint.

[LI2] Kin Y. Li, *Interpolating Blaschke Products and the left Spectrum of Multiplication Operators on the Bergman Space*, Hokkaido Mathematical Journal, Vol. 21, pp. 295-304, 1992.

[RAN] Thomas Ransoford, *Potential Theory in the Complex Plane*, Cambridge University Press, 1995.

[RIE] F. Riesz, *Über Potenzreihen mit vorgeschrieben Anfangsglieder*, Acta Math, p. 42.

[RUD] Walter Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hil Book Co., Singapore, 1987.

[VIN] S.A. Vinogradov, *Some Remakrs on Free Interpolation by Bounded and Slowly Growing Analytic Functions*, Zap. Naucn. Sem. Leningrad. Otdel. Math. Insti. Steklov (LOMI) 126, pp. 35-36, 1983.