Example 1 (2003 Spring). Let P be a countable set of points in $\mathbb{R}^2$. Prove that there exists a circle C with the origin as center and positive radius such that every point of the circle C is not in P.

(Note points inside the circle do not belong to the circle)

Sol Note that the problem is not trivial since countable set in $\mathbb{R}^2$, as in that in $\mathbb{R}$, can be very "dense" (imagine how $\mathbb{Q}\times\mathbb{Q}$ is distributed in $\mathbb{R}^2$).

Let's denote for $r \geq 0$,

$$C_r = \{(x,y) : \sqrt{x^2+y^2} = r\}.$$

In other words, C_r is a circle of radius r centered at $(0,0)$ and $C_0 = \{0\}$.

Method 1. In fact the set of circles with nonempty intersection with P must be countable. Indeed, let $P = \{\vec{p}_1, \vec{p}_2, \dots\}$ and let

$$I = \{r : C_r \cap P \neq \emptyset\}.$$

Suppose $x \in I$, then $C_x \cap P \neq \emptyset$, so there is an i such that $\vec{p}_i \in C_x$, thus $x = \|\vec{p}_i\|$ for some i, therefore the implication means the following set inclusion

$$I \subseteq \{\|\vec{p}_i\| : i = 1,2,3,\dots\}.$$

As $\{\|\vec{p}_i\| : i = 1,2,3,\dots\}$ is countable, so is I.

As $(0,\infty)\setminus I$ is uncountable, hence nonempty, thus there is an $r > 0$, $r \notin I$, i.e., $C_r \cap P = \emptyset$. ∎

Method 2. We have $\mathbb{R}^2 = \bigcup_{r\geq 0} C_r$, since every point $(x,y) \in \mathbb{R}^2$ must lie in some C_r, namely, $C_{\sqrt{x^2+y^2}}$. Thus C_r's give a decomposition of $\mathbb{R}^2$, and therefore give a decomposition of P because

$$P = P \cap \mathbb{R}^2 = P \cap \left(\bigcup_{r\geq 0} C_r\right) = \bigcup_{r\geq 0}(P \cap C_r).$$

We want to show that $P \cap C_r = \emptyset$ for some $r > 0$. **Suppose not**, then $P \cap C_r \neq \emptyset$ for every $r > 0$. Which means we can pick an element $a_r \in P \cap C_r$ for each $r > 0$.

Note that $\{a_r : r > 0\}$ is uncountable since these are extracted from each of uncountably many circles

Put in other way, the function $f : (0,\infty) \to \{a_r : r > 0\}$ given by $f(r) = a_r$ is bijective. The surjectivity follows from the way we parametrize the set $\{a_r : r > 0\}$; injectivty follows from $x \neq y \implies a_x \neq a_y$ in view of radius.

Now $P = \bigcup_{r\geq 0}(P\cap C_r) \supseteq \bigcup_{r>0}(P\cap C_r) \supseteq \bigcup_{r>0}\{a_r\} = \{a_r : r > 0\}$, contradicting the countability of P. Therefore $P \cap C_r = \emptyset$ for some $r > 0$. ∎

Example 2. Find the supremum of

(a) $A = \left\{ \dfrac{\alpha m + \beta n}{m+n} : m,n \in \mathbb{N}, m+n \neq 0 \right\}$, $\alpha, \beta > 0$.

(b) $B = \left\{ \sqrt{n} - [\sqrt{n}] : n \in \mathbb{N} \right\}$;

Sol **(a)** Suppose that $\alpha \geq \beta$, then

$$\frac{\alpha m + \beta n}{m+n} \leq \frac{\alpha(m+n)}{m+n} = \alpha$$

for every $m,n \in \mathbb{N}$. Thus A is bounded above by α. Fix $n = 1$, then

$$A \ni \frac{\alpha m + \beta}{m+1} = \frac{\alpha m}{m+1} + \frac{\beta}{m+1} \to \alpha$$

as $m \to \infty$, thus $\sup A = \alpha$.

Similarly, if $\beta \geq \alpha$, then $\sup A = \beta$. Thus actually $\sup A = \max\{\alpha, \beta\}$.

(b) Recall that for every $x \in \mathbb{R}$,

$$[x] \leq x < [x] + 1,$$

therefore

$$0 \leq x - [x] < 1.$$

It follows that B is bounded above by 1.

Consider $n = k^2 - 1$. It looks very similar to k^2 because $k^2 - 1 = k^2(1 - 1/k^2)$. It is tempting to expect

$$k - 1 < \sqrt{k^2-1} \ (< k \text{ obviously}). \tag{1}$$

If this is true, then immediately $[\sqrt{k^2-1}] = k - 1$.

To show (1), note that

$$k - 1 < \sqrt{k^2-1} \iff k^2 - 2k + 1 < k^2 - 1 \iff 2 < 2k \iff 2 \leq k.$$

Therefore (1) holds whenever $k \geq 2$, so $[\sqrt{k^2-1}] = k - 1$ whenever $k \geq 2$.

Now for $k \geq 2$,

$$\begin{aligned} B \ni \sqrt{k^2-1} - [\sqrt{k^2-1}] &= \sqrt{k^2-1} - (k-1) \\ &= \frac{k^2 - 1 - (k-1)^2}{\sqrt{k^2-1} + k - 1} \\ &= \frac{2k-2}{\sqrt{k^2-1}+k-1} = \frac{2 - \frac{2}{k}}{\sqrt{1 - \frac{1}{k^2}} + 1 - \frac{1}{k}} \to 1, \end{aligned}$$

therefore $\sup B = 1$. ∎

Remark. Actually $\{\sqrt{n}-[\sqrt{n}] : n \geq 1\}$ is dense in $[0,1]$. The construction is as follows: Let $a \in (0,1)$, then

$$\sqrt{[n^2+2an]}-[\sqrt{[n^2+2an]}] \to a,$$

and the reason behind the convergence follows from the following estimates via Taylor expansion: as $n \to \infty$,

$$\sqrt{[n^2+2an]} \leq \sqrt{n^2+2an} = n\sqrt{1+\frac{2a}{n}} = n+a+O(1/n),$$

and also

$$\sqrt{[n^2+2an]} \geq \sqrt{n^2+2an-1} = n+a+O(1/n).$$

The first estimate tells us $[\sqrt{[n^2+2an]}] = n$, combining with the second estimate we have

$$\sqrt{[n^2+2an]}-[\sqrt{[n^2+2an]}] = a+O(1/n) \tag{2}$$

as $n \to \infty$.

For example, take $a = 0.75$, then $2a = 1.5$, and

$$\sqrt{[(10^{10})^2+1.5\times 10^{10}]}-[\sqrt{[(10^{10})^2+1.5\times 10^{10}]}] = 0.749999718... \approx 0.75.$$

Example 3. Let $p \in (0,1]$, show that for every $x, y \geq 0$,

$$|x^p - y^p| \leq |x-y|^p$$

without any use of calculus.

Sol W.l.o.g. assume $y \leq x$, then we need to show

$$x^p - y^p \leq (x-y)^p \iff \left(1-\left(\frac{y}{x}\right)\right)^p \leq \left(1-\left(\frac{y}{x}\right)\right)^p \Longleftarrow 1-u^p \leq (1-u)^p, \forall u \in [0,1].$$

It is enough to show the rightmost statement.

Note that for any $x \in [0,1]$, $x \leq x^p$, it is simply because

$$x \leq x^p \iff 0 \leq x^p(1-x^{1-p})$$

and the latter holds because both $x^p \geq 0$ and $1-x^{1-p} \geq 0$ (since $1-p \geq 0$).

By using this observation, we have for every $u \in [0,1]$,

$$u \leq u^p$$

and since $1-u \in [0,1]$,

$$1-u \leq (1-u)^p,$$

we add them up to get

$$1 = u+(1-u) \leq u^p + (1-u)^p,$$

which becomes $1-u^p \leq (1-u)^p$. ∎

Example 4. Let $a_1, a_2, \cdots \in \mathbb{R}$ be such that $\lim_{n\to\infty} a_n = 1$, prove that

$$\lim_{n\to\infty}\left(\frac{a_n^{2/3}-1}{a_n-\frac{1}{2}}+\frac{n^2}{n^2+2014}\right)=1$$

by checking the definition of limit of a sequence. *Do not* use computation formulas, sandwich theorem or L'Hopital's rule.

Sol By observation we have $\frac{a_n^{2/3}-1}{a_n-\frac{1}{2}} \to \mathbf{0}$ and $\frac{n^2}{n^2+2014} \to 1$, therefore to prove convergence by definition, we split the terms in the following way:

$$L_n := \left|\frac{a_n^{2/3}-1}{a_n-\frac{1}{2}}+\frac{n^2}{n^2+2014}-1\right| \le \left|\frac{a_n^{2/3}-1}{a_n-\frac{1}{2}}\mathbf{-0}\right| + \left|\frac{n^2}{n^2+2014}-1\right| \le \frac{|a_n-1|^{2/3}}{|a_n-\frac{1}{2}|}+\frac{2014}{n}.$$

As $a_n \to 1$, we expect for large n,

$$\underbrace{\left|a_n-\frac{1}{2}\right|}_{\approx\frac{1}{2}} > \frac{1}{4}. \tag{3}$$

To show this, consider a fixed quantity $\frac{1}{4}$. By the definition of $a_n \to 1$, there is an N_1 such that

$$n > N_1 \implies |a_n - 1| < \frac{1}{4}.$$

It follows that by triangle inequality $|x-y| \ge ||x|-|y||$,

$$n > N_1 \implies \left|a_n-\frac{1}{2}\right| = \left|a_n-1+\frac{1}{2}\right| \ge \left||a_n-1|-\frac{1}{2}\right| \ge \frac{1}{2}-|a_n-1| \ge \frac{1}{2}-\frac{1}{4}=\frac{1}{4}.$$

Remark. Why we choose $\frac{1}{4}$ in (3)? Can other constants work? Let's generalize the idea to find positive lower bound instead of $\frac{1}{4}$ in (3).

Let's fix a $\delta > 0$ (supposed to describe the "closeness" of a_n to 1), then there is an $\boldsymbol{N_1}$ such that

$$n > N_1 \implies |a_n-1| < \delta.$$

From this, we have

$$\left|a_n-\frac{1}{2}\right| = \left|a_n-1+\frac{1}{2}\right| \ge \frac{1}{2}-|a_n-1| > \frac{1}{2}-\delta. \tag{3'}$$

Any $\delta > 0$ such that $\frac{1}{2}-\delta > 0 \Leftrightarrow \delta < \frac{1}{2}$ will be a sufficiently good lower bound, e.g., we may take $\delta = 1/2.00001$. The choice $\delta = \frac{1}{4} < \frac{1}{2}$ is taken simply because it looks better. Also, (3′) can be used to replace (3) in the argument.

Now fix an $\epsilon > 0$, then there is an N_2 such that

$$n > N_2 \implies |a_n-1| < \epsilon^{3/2},$$

and also by Archimedean principle, there is an N_3 such that

$$n > N_3 \implies \frac{1}{n} < \epsilon,$$

therefore

$$\begin{aligned} L_n &\le \frac{|a_n-1|^{2/3}}{|a_n-\frac{1}{2}|}+\frac{2014}{n} \\ n > \max\{N_1,N_2,N_3\} \implies \quad &\le \frac{(\epsilon^{3/2})^{2/3}}{\frac{1}{4}}+2014\epsilon \\ &= 2018\epsilon. \end{aligned}$$

∎

Example 5. Let $\{x_k\}$ converge and define $y_k = k(x_k - x_{k-1})$ for $k \geq 2$. Is $\{y_k\}$ necessarily convergent? If $\{y_k\}$ converges, show that $y_k \to 0$.

Sol $\{y_n\}$ may not converge. To see this, note

$$\sum_{k=2}^{n} \frac{y_k}{k} = \sum_{k=2}^{n}(x_k - x_{k-1}) = x_n - x_1,$$

for x_n to be convergent, we may take $y_k = (-1)^k$, then $\{x_n\}$ converges by Alternating Series Test, but $\{y_k\} = \{(-1)^k\}$ is divergent.

Suppose now $y_n \to \ell$, we show $\ell = 0$. Suppose not, then we have either two cases:

Case 1. $\ell > 0$, in this case, we can find an N such that

$$k > N \implies y_k > \frac{\ell}{2},$$

it follows that

$$x_n - x_1 = \sum_{k=2}^{N} \frac{y_k}{k} + \sum_{k=N+1}^{n} \frac{y_k}{k} > \sum_{k=2}^{N} \frac{y_k}{k} + \frac{\ell}{2} \sum_{k=N+1}^{n} \frac{1}{k},$$

then $x_n \to \infty$ by taking $n \to \infty$, a **contradiction** to that $\{x_n\}$ is convergent.

Case 2. $\ell < 0$, then there is an N such that

$$k > N \implies y_k < \frac{\ell}{2},$$

it follows that

$$x_n - x_1 = \sum_{k=2}^{N} \frac{y_k}{k} + \sum_{k=N+1}^{n} \frac{y_k}{k} < \sum_{k=2}^{N} \frac{y_k}{k} + \frac{\ell}{2} \sum_{k=N+1}^{n} \frac{1}{k},$$

then $x_n \to -\infty$ by taking $n \to \infty$, again the same **contradiction**. ∎

Example 6. Suppose $x_1, x_2, \cdots \geq 0$ and $\lim_{n\to\infty} (-1)^n x_n$ exists, show that $\lim_{n\to\infty} x_n$ also exists.

Sol Let $\ell = \lim_{n\to\infty} (-1)^n x_n$, by considering even and odd indexes (two subseqs of $\{(-1)^n x_n\}$, which must be convergent), then we have

$$\lim_{n\to\infty} x_{2n} = \ell = \lim_{n\to\infty} (-1)x_{2n-1}. \qquad (\heartsuit)$$

Let

$$\lim_{n\to\infty} x_{2n} = a \quad \text{and} \quad \lim_{n\to\infty} x_{2n-1} = b,$$

then $(\heartsuit)$ becomes $a = \ell = -b$, this says that $a + b = 0$.

Since $x_n \geq 0$ for every n, we have $a, b \geq 0$, therefore $a + b = 0 \implies a = b = 0$, showing that $\lim_{n\to\infty} x_n = 0$. ∎