A note on some supercongruences

Abstract

The purpose of this paper is to provide an elementary proof via the theory of hypergeometric series for some supercongruences results.

1. Introduction

In an interesting article ON SUMS OF APÉRY POLYNOMIALS AND RELATED CONGRUENCES written by Zhi Wei Sun there are two congruences:

Theorem 1. *Let p be an odd prime and $k \in \{0, 1, \ldots, p-1\}$. If $x \equiv k \pmod p$, then*

$$\sum_{r=0}^{p-1} \binom{x}{r}^2 \equiv \binom{2x}{k} \pmod{p^2}.$$

Theorem 2. *Let p be an odd prime, $x \equiv 2k \pmod p$ and $k \in \{0, \ldots, \frac{p-1}{2}\}$, then*

$$\sum_{r=0}^{p-1} (-1)^r \binom{x}{r}^2 \equiv (-1)^k \binom{x}{k} \pmod{p^2}.$$

In his proof for theorem 2 he first defined for $k \in \mathbb{N}$, $f_k(y) := \sum_{r=0}^{p-1} (-1)^r \binom{2k+py}{r}^2$, and then used the Zeilberger algorithm via mathematica 7 to generate a recurrence relation $Af_{k+1}(y) + Bf_k(y) = C\binom{py+2k+2}{p-1}^2$, where A, B and C are functions in p, y and k, from that a careful expansion and computation result in theorem 2. And he mentioned theorem 1 follows from a similar approach. We see that the use of computer algebra is a sort of brute force and what Zhi Wei Sun has done in his paper is nearly impossible to be achieved by human. In what follows we try to apply some results on special function which do help us prove the above two congruences in a “comfortable way”.

2. Proof of Theorem 1 and 2

We start with providing two identities in the theory of hypergeometric series. Here $\Gamma(z)$ denotes the Gamma-function defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

and a, b, c are complex numbers which are such that the singularity of Γ is not reached.

Theorem 3 (Gauss's Hypergeometric Theorem). *When* $\mathrm{Re}(c-a-b) > 0$, *then the following is true:*

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Theorem 4 (Kummer's Theorem). *When* $\operatorname{Re} b < 1$, *then the following holds:*

$$ {}_2F_1\left[\begin{matrix} a, b \\ 1+a-b \end{matrix}; -1\right] = \frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}. $$

Proof. (of theorem 1) We see that theorem theorem holds immediately when $k=0$. Hence we now assume $k \geq 1$. Define $f(z) = \sum_{r=0}^{p-1} \binom{z}{r}^2 = \sum_{r=0}^{p-1} \frac{(-z)_r^2}{r!^2}$. As $x = k + lp$ for some $l \in \mathbb{Z}$ and f is a polynomial with at most degree $2p$,

$$ f(x) = f(k+lp) = f(k) + f'(k)lp + \sum_{n=2}^{2p} \frac{f^{(n)}(k)}{n!}(lp)^n. \tag{1} $$

We can see that $\frac{f^{(n)}(k)}{n!}(lp)^n \equiv 0 \pmod{p^2}$ for all $n \geq 2$, the reason is that $f^{(n)}(k)$ is always congruent to an integer a_n modulo p^2 and $\frac{1}{n!}(lp)^n \equiv 0 \pmod{p^2}$ when $n \geq 2$ (as when $n=2$, no factor p in denominator while when $n \geq 3$, the number of factor p is at least $n - \lfloor \frac{n}{p} \rfloor \geq n - \lfloor \frac{n}{3} \rfloor = 2 \cdot \frac{n}{3} + \frac{n}{3} - \lfloor \frac{n}{3} \rfloor \geq 2$). So from (1),

$$ f(x) = f(k+lp) \equiv f(k) + f'(k)lp \pmod{p^2}. \tag{2} $$

Since $f(k) = \sum_{r=0}^{p-1} \frac{(-k)_r^2}{r!^2} = \sum_{r=0}^{\infty} \frac{(-k)_r^2}{r!^2} = {}_2F_1\left[\begin{matrix} -k, -k \\ 1 \end{matrix}; 1\right]$, by theorem 3, we have

$$ f(k) = {}_2F_1\left[\begin{matrix} -k, -k \\ 1 \end{matrix}; 1\right] = \frac{\Gamma(1)\Gamma(1+2k)}{\Gamma(1+k)^2} = \frac{(2k)!}{k!k!} = \binom{2k}{k}. $$

It remains to evaluate $f'(k)$,

$$ f'(k) = \sum_{r=0}^{p-1} 2\binom{x}{r}\frac{d}{dx}\binom{x}{r}\Bigg|_{x=k} = \sum_{r=0}^{\infty} 2\binom{x}{r}\frac{d}{dx}\binom{x}{r}\Bigg|_{x=k} = \frac{d}{dx}\underbrace{\sum_{r=0}^{\infty}\binom{x}{r}^2}_{:=F(x)}\Bigg|_{x=k} = F'(k), $$

theorem 3 tells us when $x > 0$, $\operatorname{Re}(1-(-x)-(-x)) = 1 + 2x > 0$, thus $F(x) = {}_2F_1\left[\begin{matrix} -x, -x \\ 1 \end{matrix}; 1\right] = \frac{\Gamma(1+2x)}{\Gamma(1+x)^2}$. Differentiating F once and by the formula $F'(1+n) = n!\big(1 + \frac{1}{2} + \cdots + \frac{1}{n} + \Gamma'(1)\big)$ for $n \in \mathbb{N}$, we obtain

$$ \begin{aligned} &F'(k) \\ &= 2\Gamma(1+x)^{-3}\Big(-\Gamma(1+2x)\Gamma'(1+x) + \Gamma'(1+2x)\Gamma(1+x)\Big)\Big|_{x=k} \\ &= 2(k!)^{-3}\left(-(2k)!(k!)\left(1+\frac{1}{2}+\cdots+\frac{1}{k}+\Gamma'(1)\right) + (2k)!(k!)\left(1+\frac{1}{2}+\cdots+\frac{1}{2k}+\Gamma'(1)\right)\right) \\ &= 2(k!)^{-2}(2k)!\left(\frac{1}{k+1}\cdots+\frac{1}{2k}\right) = 2\binom{2k}{k}\sum_{r=k+1}^{2k}\frac{1}{r}. \end{aligned} $$

We continue from (2) and combine results above,

$$ f(x) \equiv \binom{2k}{k} + 2lp\binom{2k}{k}\sum_{r=k+1}^{2k}\frac{1}{r} \pmod{p^2}. \tag{3} $$

On the other hand as $x = k + lp$,

$$\begin{aligned}
\binom{2x}{k} &= \frac{1}{k!}\prod_{r=0}^{k-1}(2x-r) = \frac{1}{k!}\prod_{r=0}^{k-1}(2k-r+2lp) \\
&\equiv \frac{1}{k!}\left(\prod_{r=0}^{k-1}(2k-r) + 2lp\Big(\prod_{r=0}^{k-1}(2k-r)\Big)\Big(\sum_{r=0}^{k-1}\frac{1}{2k-r}\Big)\right) \pmod{p^2} \\
&\equiv \frac{(2k)!}{k!k!}\left(1 + 2lp\sum_{r=k+1}^{2k}\frac{1}{r}\right) \pmod{p^2},
\end{aligned}$$

so

$$\binom{2x}{k} \equiv \binom{2k}{k} + 2lp\binom{2k}{k}\sum_{r=k+1}^{2k}\frac{1}{r} \pmod{p^2}, \tag{4}$$

thus (4) and (3) give desired result.

Proof. (of theorem 2) The case that $k = 0$ is immediately true. Let's assume now $k \geq 1$. Define $f(z) = \sum_{r=0}^{p-1}(-1)^r\binom{z}{r}^2$, write $x = 2k + lp$, for some $l \in \mathbb{Z}$, then

$$f(x) = f(2k+lp) \equiv f(2k) + f'(2k)lp \pmod{p^2}. \tag{5}$$

The reasoning for the above to be true is similar to that before we arrive to (2). We evaluate $f(2k)$ first, since $(-1)^r\binom{z}{r}^2 = (-1)^r\left(\frac{(-1)^r(-z)_r}{r!}\right)^2 = \frac{(-z)_r^2}{(1)_r}\frac{(-1)^r}{r!}$, thus

$$f(2k) = \sum_{r=0}^{\infty}\frac{(-2k)_r^2}{(1)_r}\frac{(-1)^r}{r!} = \underbrace{\sum_{r=0}^{\infty}\frac{(-z)_r^2}{(1)_r}\frac{(-1)^r}{r!}}_{:=F(z)}\Bigg|_{z=2k} = F(2k).$$

Moreover, assuming z is taken nice enough such that the following holds (that is, the singularity of Γ is not reached), by theorem 4,

$$\begin{aligned}
F(z) = {}_2F_1\left[\begin{matrix}-z,-z \\ 1\end{matrix};-1\right] &= \frac{\Gamma(1)\Gamma(1-\frac{1}{2}z)}{\Gamma(1-z)\Gamma(1-\frac{1}{2}z+z)} \\
&= \frac{\Gamma(1-\frac{z}{2})}{\Gamma(1-z)\Gamma(1+\frac{z}{2})}.
\end{aligned}$$

Before we take $z = 2k$ ($k \geq 1$) we use the formula $\Gamma(1-z) = -\frac{\pi}{\Gamma(z)\sin\pi z}$ to get rid of the term $\Gamma(1-\frac{z}{2})$ and $\Gamma(1-z)$ for which $z = 2k$ is a singularity, thus

$$F(z) = \frac{\pi}{\Gamma(\frac{z}{2})\sin\frac{\pi}{2}z}\cdot\frac{\Gamma(z)\sin 2(\frac{\pi}{2}z)}{\pi}\cdot\frac{1}{\Gamma(1+\frac{z}{2})} = 4\frac{\Gamma(z)\cos z\frac{\pi}{2}}{z\Gamma(\frac{z}{2})^2}, \tag{6}$$

the last equality follows from $\Gamma(z+1) = z\Gamma(z)$. Due to the expression on RHS of (6), $F(z)$ is now continuous at $2k$ and hence by taking limit on both sides of this equation, we get

$$f(2k) = F(2k) = \frac{2\Gamma(2k)\cos k\pi}{k\Gamma(k)^2} = (-1)^k\binom{2k}{k}. \tag{7}$$

We now compute $f'(2k)$.

$$\begin{aligned} f'(2k) &= \sum_{r=1}^{p-1} (-1)^r 2\binom{z}{r}\frac{d}{dz}\binom{z}{r}\bigg|_{z=2k} \\ &= \sum_{r=1}^{\infty} (-1)^r 2\binom{z}{r}\frac{d}{dz}\binom{z}{r}\bigg|_{z=2k} \\ &= \frac{d}{dz}\underbrace{\sum_{r=0}^{\infty}(-1)^r\binom{z}{r}^2}_{\text{defined to be } F(z)}\bigg|_{z=2k} = F'(2k), \end{aligned}$$

so we can make use of the expression of $F(z)$ in (6), let's do direct differentiation.

$$\begin{aligned} F'(2k) = F'(z)\Big|_{z=2k} &= 4\cos\left(\frac{z\pi}{2}\right)\cdot\frac{d}{dz}\left(\frac{\Gamma(z)}{z\Gamma(\frac{z}{2})^2}\right)\bigg|_{z=2k} \\ &= 4\cos k\pi \frac{z\Gamma(\frac{z}{2})^2\Gamma'(z) - \Gamma(z)\Big(z\Gamma(\frac{z}{2})\Gamma'(\frac{z}{2}) + \Gamma(\frac{z}{2})^2\Big)}{z^2\Gamma(\frac{z}{2})^4}\Bigg|_{z=2k}. \end{aligned}$$

Finally by the formula $\Gamma(1+n) = n!$ (n nonnegative integer) and $\Gamma'(1+k) = k!\big(1+\frac{1}{2}+\cdots+\frac{1}{k}+\Gamma'(1)\big)$, we simplify the above to get

$$F'(2k) = (-1)^k\binom{2k}{k}\sum_{r=k+1}^{2k}\frac{1}{r},$$

combining this result with (5) and (7), we deduce the following

$$f(x) \equiv (-1)^k\binom{2k}{k}\left(1 + lp\sum_{r=k+1}^{2k}\frac{1}{r}\right) \pmod{p^2}.$$

Finally we will show that RHS of the congruence in the proposition share the same congruence modulo p^2, this is achieved by direct expansion of the product modulo p^2 as follows:

$$\begin{aligned} (-1)^k\binom{x}{k} &= (-1)^k\binom{2k+lp}{k} \\ &= (-1)^k\frac{1}{k!}\prod_{r=0}^{k-1}(2k-r+lp) \\ &\equiv (-1)^k\frac{1}{k!}\left(\prod_{k=0}^{k-1}(2k-r) + \prod_{r=0}^{k-1}(2k-r)\sum_{r=0}^{k-1}\frac{1}{2k-r}lp\right) \pmod{p^2} \\ &\equiv (-1)^k\binom{2k}{k}\left(1 + lp\sum_{r=k+1}^{2k}\frac{1}{r}\right) \pmod{p^2}, \end{aligned}$$

as was to be shown.