Orthogonal Projection onto Finite Dimensional Subspaces;
Minimizer Obtained by Orthogonal Projections

Key Questions

- What is orthogonal projection?
- Let $P_W : V \to W$ be orthogonal projection onto W (finite dimensional!), what properties do $P_W v$ have?
- How to use the concept of orthogonal projection to solve minimizing problems in linear algebra?

Problem 1. Let V be an inner product space. Suppose W is a finite dimensional subspace of V, let's choose α and β be two orthogonal bases of W, then we can perform orthogonal projection onto W by using either one of bases.

> **Question:** Explain why are the orthogonal projections constructed by using α and β onto W the same?

Remark. Which means that whenever we obtain an orthogonal basis of a subspace, we can freely choose this basis to perform orthogonal projection. Choice is not a matter ☺.

Solution.

Problem 2. Let $y = \begin{bmatrix} 7 \\ 4 \\ 7 \end{bmatrix}$ and let V be a subspace of $\mathbb{R}^3$ spanned by

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad u_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

(a) Find a $u \in V$ that is closest to y by orthogonal projection.

(b) Find a $u \in V$ that is closest to y by constructing a **normal equation**.

(c) Find a $u \in V$ that is closest to y by property of minimizing element.

(d) Construct a matrix $P \in M_{3\times 3}(\mathbb{R})$ such that

$$\|x - Px\| \leq \|x - v\|$$

for all $x \in \mathbb{R}^3$ and $v \in V$. What is Py?

Solution.

Problem 3. Find $p \in \mathbb{P}_3$ such that $p(0) = 0, p'(0) = 0$ and

$$\int_0^1 (2 + 3x - p(x))^2 \, dx$$

is as small as possible.

Solution.

Problem 4. Let $A \in M_{m\times n}(\mathbb{R})$, prove that A is injective if and only if A^T is surjective.

Solution.

Problem 5. Let $m \leq n$ and $b, a_1, \ldots, a_m \in \mathbb{R}^n$. Show that if

$$a_1 \cdot v = a_2 \cdot v = \cdots = a_m \cdot v = 0 \implies b \cdot v = 0, \quad \text{for every } v \in \mathbb{R},$$

then b is a linear combination of a_i's.

Solution.

Problem 6. Let $A \in M_{n\times n}(\mathbb{R})$ be such that $A^2 = A$. Prove that the following are equivalent:

(i) A is an orthogonal projection (i.e., $\operatorname{Nul} A \perp \operatorname{Col} A$).

(ii) A is symmetric (i.e., $A^T = A$).

Solution.