Math2121 (Spring 2012-2013) **Tutorial Note 9**

Inner Product, "Norm" (i.e., "Length");
Orthogonal and Orthonormal Sets, Orthogonal Complement;
Gram-Schmidt Orthogonalization Process

Key Questions

- What is an inner product on a vector space?
- What do we mean by orthogonal set and orthonormal set?
- What is orthogonal complement? What properties does it have?
- What is Gram-Schmidt orthogonalization process?

Problem 1. For $(a_1, a_2, \ldots, a_n)^T, (b_1, b_2, \ldots, b_n)^T \in \mathbb{R}^n$, we have the celebrated Cauchy-Schwarz inequality:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

This follows easily from the fact that dot product is an inner product on $\mathbb{R}^n$. Prove that all roots of $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ lie in the open disk

$$\{z \in \mathbb{C} : |z| < \sqrt{1 + |a_{n-1}|^2 + \cdots + |a_1|^2 + |a_0|^2}\}.$$

Solution.

Problem 2. Let $\langle \cdot, \cdot \rangle$ be a function on $M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R})$ given by

$$\langle A, B \rangle = \text{Tr}(B^T A).$$

(a) Show that $\langle \cdot, \cdot \rangle$ is an inner product.

(b) In particular, consider $n = 2$. Let $M_{2\times 2}(\mathbb{R})$ have the inner product defined above, determine if

$$S = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & 0 \end{bmatrix} \right\}$$

is orthogonal.

Solution.

Problem 3. Show that on $C[a,b]$,

$$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$$

defines an inner product.

Solution.

Problem 4.

(a) Find an orthogonal basis of $\mathbb{P}|_{[0,1]} := \{p|_{[0,1]} : p \in \mathbb{P}_2\}$ with inner product defined by $\langle p,q\rangle = \int_0^1 pq\,dx$.

(b) From (a), construct an orthogonal basis of $\mathbb{P}|_{[-\pi,\pi]} := \{p|_{[-\pi,\pi]} : p \in \mathbb{P}_2\}$ with inner product defined by $\langle p,q\rangle = \int_{-\pi}^{\pi} pq\,dx$.

Solution.

Problem 5. Let V be an inner product space, prove that $\langle u, v\rangle = 0$ if and only if $\|u\| \leq \|u + av\|$ for every $a \in \mathbb{R}$.

Solution.

Problem 6.

(a) Let V be a vector space and $S \subseteq V$, with $0 \in S$, prove that

$$S \cap S^{\perp} = \{0\}.$$

Also, if W_1, W_2 are subspaces of V, show that

$$(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}.$$

(b) **(For Enrichment Only)** If W is a **finite dimensional** subspace of V, we know that $(W^{\perp})^{\perp} = W^{(*)}$, give an example of an infinite dimensional vector space V whose subspace W satisfies $(W^{\perp})^{\perp} \neq W$.

Solution.

(*) This can be easily explained by orthogonal projection in the next tutorial.