Eigenvalues and Eigenvectors for Square Matrices;
Algebraic and Geometric Multiplicity of Eigenvalues; Diagonalization

Key Questions

- What are eigenvalue, eigenspace and eigenvector?
- What is characteristic polynomial of a matrix?
- Given an eigenvalue λ of a square matrix, what do we mean by algebraic multiplicity of λ?
- What is the meaning of diagonalizability? What is its relation with eigenvalues?

Problem 1. What is the characteristic polynomial of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$?

Solution.

Problem 2. Prove that if

$$A = \begin{bmatrix} a_{11} & & & * \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix},$$

then $a_{11}, a_{22}, \ldots, a_{nn}$ are all eigenvalues of A.

Solution.

Problem 3. Let $A = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}$, find all eigenvalues and eigenvectors of A.

Solution.

Problem 4. Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

find the algebraic and geometric multiplicities of all eigenvalues.

Solution.

Problem 5. Is $\begin{bmatrix} 1 & 6 & 5 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$ diagonalizable? How about $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$?

Solution.

Problem 6. Is $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ diagonalizable? If it does, find its diagonalization.

Solution.

Problem 7. Let $A \in M_{n\times n}(\mathbb{R})$, show that for any $\epsilon > 0$, there is t with $0 < |t| < \epsilon$ such that $A - tI$ is invertible.

Remark. From that the equality

$$\det(I + AB) = \det(I + BA)$$

easily follows, where A, B are matrices such that AB and BA are square and I is "an" identity matrix that is of appropriate size (two symbols I means different identity matrices). If one is interested, first assume that A, B are both square, and then extend the result to general case.

Solution.

Problem 8. Let $A \in M_{m\times n}(\mathbb{R})$ and $\dim \operatorname{Col} A = k$, show that A has at most $k+1$ distinct eigenvalues.

Solution.