Math2121 (Spring 2012-2013) **Tutorial Note 7**

Coordinates and Matrix Representations of Linear Transformations; Generalized Nullity-Rank Theorem

Key Questions

- What is the coordinate vector of a vector w.r.t. a basis?
- What is the matrix representation of a linear transformation when bases are given?
- What is the general version of nullity-rank theorem?
- What is a transition matrix from one basis to another basis?

Problem 1. Let $A = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$ and $\beta = \left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix} \right\}$. We have shown that A is invertible and $A^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ in tutorial note 4, find the coordinate vector of $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ w.r.t. β.

Solution.

Problem 2. Let the linear map $T : \mathbb{P}_2 \to \mathbb{P}_4$ be defined by

$$(Tp)(x) = \frac{dp}{dx}(x) + x^2 p(x).$$

Let $\alpha = \{1, x, x^2\}$ and $\beta = \{1, x, x^2, x^3, x^4\}$, find $[T]_\alpha^\beta$. Is T injective?

Solution.

Problem 3. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, define a linear map $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by

$$T(A) = MA.$$

Find $[T]_\beta$, where

$$\beta = \left\{ E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Solution.

Problem 4. Let $\alpha = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ and $\beta = \left\{ \begin{bmatrix} 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 9 \\ 10 \end{bmatrix} \right\}$, find the transition matrix from α to β.

Solution.

Problem 5. Let $B \in M_{n\times n}(\mathbb{R})$. Suppose $B^k = 0$ for some $k \in \mathbb{N}$, show that every matrix in $M_{n\times n}(\mathbb{R})$ has the form $BA - A$, for some $A \in M_{n\times n}(\mathbb{R})$.

Solution.

Problem 6 (Interpolation Problem). Let $a_1, a_2, \ldots, a_{n+1}$ be $n+1$ distinct numbers on the x-axis. Given $b_1, b_2, \ldots, b_{n+1} \in \mathbb{R}$, show that there is a polynomial $p \in \mathbb{P}_n$ such that

$$p(a_1) = b_1, \quad p(a_2) = b_2, \quad \ldots, \quad p(a_{n+1}) = b_{n+1}.$$

Solution.

Definition. Let V be a real vector space, the vector space

$$V^* := \mathcal{L}(V, \mathbb{R})$$

of all linear maps from V to $\mathbb{R}$ is called the **dual space** of V.

Basic Fact. When $\dim V < \infty$, we can give V a basis $\{v_1, v_2, \ldots, v_k\}$, then the linear maps $v_1^*, v_2^*, \ldots, v_k^*$ defined by

$$v_i^*(v_j) = \delta_{ij}, \quad j = 1, 2, \ldots, k$$

form a basis of V^*. Hence $\dim V = \dim V^*$.

Problem 7. Let $M_{n\times n}(\mathbb{R})$ denote the vector space of all $n \times n$ matrices. For every $C \in M_{n\times n}(\mathbb{R})$, define the linear map $T_C : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ in the following way

$$T_C(A) = \mathrm{Tr}(CA), \quad \text{for each } A \in M_{n\times n}(\mathbb{R}).$$

It is clear that $T_C \in (M_{n\times n}(\mathbb{R}))^*$, show that actually,

$$\left(M_{n\times n}(\mathbb{R})\right)^* = \{T_C : C \in M_{n\times n}(\mathbb{R})\}.$$

Solution.