Math2121 (Spring 2012-2013) **Tutorial Note 6**

Important Subspaces of $\mathbb{R}^n$, Row, Column Spaces;
Rank-Nullity Theorem

Key Questions

- What are $\operatorname{Nul} A$ and $\operatorname{Col} A$ for a given matrix A?
- What is the standard way to compute the basis of column space and null space of a given matrix?
- What is rank? What is nullity? What is rank-nullity theorem?

Problem 1. Let

$$A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 4 & -3 & 4 \\ 5 & 10 & -8 & 11 \end{bmatrix}$$

find $\dim \operatorname{Nul} A$.

Solution.

Problem 2. Let

$$A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 4 & -3 & 4 \\ 5 & 10 & -8 & 11 \end{bmatrix},$$

find $\dim \operatorname{Col} A$.

Solution.

Problem 3. Let A be an $n \times n$ matrix, show that the following are equivalent:

(a) A is injective.

(b) A is surjective.

(c) A is invertible.

Solution.

Problem 4. Let $S \in M_{n\times n}(\mathbb{R})$ be skew-symmetric (i.e., $S^T = -S$), show that $I + S$ is invertible.

Solution.

Problem 5. Let $A \in M_{m\times k}(\mathbb{R})$ and $B \in M_{k\times n}(\mathbb{R})$, then $AB \in M_{m\times n}(\mathbb{R})$, prove that

$$\dim \operatorname{Nul}(AB) \leq \dim \operatorname{Nul} A + \dim \operatorname{Nul} B.$$

Solution.

Problem 6. Prove that if $A \in M_{2\times 4}(\mathbb{R})$ satisfy

$$\operatorname{Nul} A = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_1 = 5x_2, x_3 = 7x_4 \right\},$$

then $x \mapsto Ax$ is surjective.

Solution.