Math2121 (Spring 2012-2013) **Tutorial Note 5**

Vector Spaces Again, Bases and Dimensions

Key Questions

- Again, what are vector spaces?
- What are finite dimensional vector spaces?
- What is a basis of a finite dimensional vector space?
- What do we mean by dimension of a vector space?
- Which vector spaces are infinite dimensional?

Problem 1. Show that $\mathbb{R}^n, M_{m\times n}(\mathbb{R}), \mathbb{P}_n$ are all finite dimensional vector spaces. What are their bases?

Solution.

Problem 2. Show that

$$\mathbb{P} = \{\text{polynomial with real coefficient(s)}\} \quad \text{and} \quad C(\mathbb{R})$$

are infinite dimensional.

Solution.

Problem 3. Find a basis of the vector space

$$\mathcal{F} = \{f : \{1,2,\ldots n\} \to \mathbb{R}\}$$

which has naturally defined addition and scalar multiplication.

Solution.

Problem 4. If U is a subspace of V and $\dim U = \dim V < \infty$, show that $U = V$. Hence, prove that for given $v_1, v_2, v_3 \in \mathbb{R}^3$, if the vector equation

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = b$$

has no solution of some $b \in \mathbb{R}^3$, then $\{v_1, v_2, v_3\}$ is linearly dependent.

Solution.

Problem 5. Let $\mathrm{Tr}[a_{ij}]_{n\times n} = a_{11} + a_{22} + \cdots + a_{nn}$, show that

$$V = \{A \in \mathbb{R}^{n\times n} : \mathrm{Tr}\, A = 0\}$$

is a subspace of $\mathbb{R}^{n\times n}$, what is its dimension?

Solution.

Problem 6. Determine which of the following sets are bases for $\mathbb{R}^3$:

(a) $\{(1,0,-1)^T,(2,5,1)^T,(0,-4,3)^T\}$ ______

(b) $\{(2,-4,1)^T,(0,3,-1)^T,(6,0,-1)^T\}$ ______

(c) $\{(1,2,-1)^T,(1,0,2)^T,(2,1,1)^T\}$ ______

(d) $\{(-1,3,1)^T,(2,-4,-3)^T,(-3,8,2)^T\}$ ______

(e) $\{(1,-3,-2)^T,(-3,1,3)^T,(-2,-10,-2)^T\}$ ______

Solution.

Problem 7. Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$, prove that

$$\operatorname{rank}(AB) \leq \min\{\operatorname{rank} A, \operatorname{rank} B\}.$$

Solution.