Linear Transformations

Key Questions

- What are linear transformations? (see page 1 of note 4)
- How to obtain a matrix of a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$?

Problem 1. In tutorial note 2 we know that $C[0,1]$ is a vector space with naturally defined addition and scalar multiplication. Show that $T : C[0,1] \to C[0,1]$ defined by

$$(Tf)(x) = \int_0^x f(t)\,dt$$

is linear. Moreover, find all the solution of $Tf = 0$. i.e., what is $\{f \in C[0,1] : Tf = 0\}$?

Solution.

Problem 2.

(a) Find a matrix R_θ that rotates a point $p \in \mathbb{R}^2$ by an angle θ counter-closckwise about the origin.

(b) Also find a matrix B that reflects a point along the diagonal $y = x$.

(c) Finally, find one that reflects a point along the line $y = \tan \alpha x$, where $\alpha \in [0, \pi/2]$.

Solution.

Problem 3. Determine which of the following maps are linear transformations.

(a) The transformation T defined by $T(x_1, x_2)^T = (2x_1 - 3x_2, x_1 + 4, 5x_2)^T$.

(b) The transformation T defined by $T(x_1, x_2)^T = (4x_1 - 2x_2, 3|x_2|)^T$.

(c) The transformation T defined by $T(x_1, x_2, x_3)^T = (1, x_2, x_3)^T$.

(d) The transformation T defined by $T(x_1, x_2, x_3)^T = (x_1, 0, x_3)^T$.

(e) The transformation T defined by $T(x_1, x_2, x_3)^T = (x_1, x_2, -x_3)^T$.

Solution.

Problem 4. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Determine whether or not T is one-to-one in each of the following situations:

______ **(a)** When $n > m$.

______ **(b)** When $n = m$.

______ **(c)** When $n < m$.

Fill the symbols A, B and C in ______ defined below:

$\boxed{\text{A}}$ T is a one-to-one transformation.

$\boxed{\text{B}}$ T is not a one-to-one transformation.

$\boxed{\text{C}}$ There is not enough information to tell.

Solution.

Problem 5. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let A be the standard matrix of T.

Fill the correct symbols A, B and C in ______ for each of the following situations.

______ **(a)** If every row in the row echelon form of A has a pivot.

______ **(b)** If the row echelon form of A has a row of zeros.

______ **(c)** If two rows in the row echelon form of A do not have pivots.

______ **(d)** If the row echelon form of A has a pivot in every column.

Where:

$\boxed{\text{A}}$ T is not onto.

$\boxed{\text{B}}$ T is onto.

$\boxed{\text{C}}$ there is not enough information to tell.

Solution.