Vector Spaces, Linear Span and Linear Independence

Key Questions

- What are vector spaces?
- What is linear span?
- What is linear independence?

Problem 1. How to define **(i)** additions and **(ii)** scalar multiplications that make the following sets into vector spaces?

(a) $\mathbb{R}^\infty := \{(a_1, a_2, a_3, \dots) : a_1, a_2, a_3, \dots \in \mathbb{R}\}$

(b) $C[a,b]$ and $\mathcal{R}[a,b]$

(c) $\mathbb{P}_n := \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 : a_0, a_1, \dots, a_n \in \mathbb{R}\}$

(d) $M_{m\times n}(\mathbb{R}) := \{[a_{ij}]_{m\times n} : a_{ij} \in \mathbb{R}, i = 1,2,\dots,m, j = 1,2,\dots,n\}$

Solution.

Problem 2. By definition, what is the span of

$$V = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}?$$

Also, is $b = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 3 \end{bmatrix}$ in the span of V?

Solution.

Problem 3. If $\{v_1, v_2, \ldots, v_n\}$ are linearly independent in a vector space V, can some of v_i's be zero vector?

Solution.

Problem 4. Is the set

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$$

linearly independent?

Solution.

Problem 5. Show that every set of $n+1$ vectors in $\mathbb{R}^n$ must be linearly dependent.

Solution.

Problem 6. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be distinct real numbers, show that $\{e^{\alpha_1 t}, e^{\alpha_2 t}, \ldots, e^{\alpha_n t}\}$ is linearly independent in $C(\mathbb{R})$.

Solution.

Problem 7 (Wronskian). Let $f_1, f_2, \ldots, f_n$ be $n-1$ times differentiable functions on their common domain. Prove that if

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}(x_0) \neq 0,$$

for some x_0, then $\{f_1, f_2, \ldots, f_n\}$ is linearly independent. As an application, show that $\{x, xe^x, x^2e^x\}$ is linearly independent.

In this problem we take the following result for granted: $\det A \neq 0$ *iff* A *is invertible. We will introduce the concept of determinants in tutorial note 5.*

Solution.