Math2121 (Spring 2012-2013)

Tutorial Note 2

Vector Spaces, Linear Span and Linear Independence

– Key Questions –

- What are vector spaces?
- What is linear span?
- What is linear independence?

Problem 1. How to define (i) additions and (ii) scalar multiplications that make the following sets into vector spaces?

(a) $\mathbb{R}^{\infty} := \{(a_1, a_2, a_3, \dots) : a_1, a_2, a_3, \dots \in \mathbb{R}\}$

(b) C[a,b] and $\Re[a,b]$

(c) $\mathbb{P}_n := \{a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 : a_0, a_1, \dots, a_n \in \mathbb{R}\}$

(d) $M_{m \times n}(\mathbb{R}) := \{ [a_{ij}]_{m \times n} : a_{ij} \in \mathbb{R}, i = 1, 2, \dots, m, j = 1, 2, \dots, n \}$

Solution.

Problem 2. By definition, what is the span of

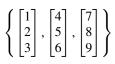
$$V = \left\{ \begin{bmatrix} 1\\0\\1\\1 \end{bmatrix}, \begin{bmatrix} 1\\2\\-1\\1\\1 \end{bmatrix}, \begin{bmatrix} 1\\-1\\1\\-1 \end{bmatrix}, \begin{bmatrix} 0\\0\\1\\-2 \end{bmatrix} \right\}?$$

Also, is $b = \begin{bmatrix} 3\\1\\0\\3 \end{bmatrix}$ in the span of V?

Problem 3. If $\{v_1, v_2, ..., v_n\}$ are linearly independent in a vector space *V*, can some of v_i 's be zero vector?

Solution.

Problem 4. Is the set



linearly independent?

Problem 5. Show that every set of n + 1 vectors in \mathbb{R}^n must be linearly dependent.

Solution.

Problem 6. Let $\alpha_1, \alpha_2, ..., \alpha_n$ be distinct real numbers, show that $\{e^{\alpha_1 t}, e^{\alpha_2 t}, ..., e^{\alpha_n t}\}$ is linearly independent in $C(\mathbb{R})$.

Problem 7 (Wronskian). Let $f_1, f_2, ..., f_n$ be n - 1 times differentiable functions on their common domain. Prove that if

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f'_1 & f'_2 & \cdots & f'_n \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} (x_0) \neq 0,$$

for some x_0 , then $\{f_1, f_2, ..., f_n\}$ is linearly independent. As an application, show that $\{x, xe^x, x^2e^x\}$ is linearly independent.

In this problem we take the following result for granted: det $A \neq 0$ iff A is invertible. We will introduce the concept of determinants in tutorial note 5.