Math3033 (Fall 2013-2014) **Tutorial Note 10**

Lebesgue Integration and Convergence Theorems

Key Definitions and Results

Definition 1. A function f is said to be **simple** if it is of the form $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ for some $a_i \in \mathbb{R}$ and measurable sets A_i.

Equivalently, $f : E \to \mathbb{R}$ is **simple** if it is measurable and takes only finitely many values in its range.

Definition 2. Given a function $f : E \to \mathbb{R}$, we call

$$f^+ = \max\{f(x), 0\} \quad \text{and} \quad f^- = \max\{-f(x), 0\}$$

the **positive part** and **negative part** of f respectively.

Definition 3. Let $f : E \to \mathbb{R}$ be measurable. If at least one of $\int_E f^+\,dm$ and $\int_E f^-\,dm$ is finite, then we define the **Lebesgue integral of f over E** by

$$\int_E f\,dm = \int_E f^+\,dm - \int_E f^-\,dm.$$

Definition 4. A function $f : E \to \mathbb{R}$ is said to be **Lebesgue integrable** if it is measurable and $\int_E |f|\,dm = \int_E f^+\,dm + \int_E f^-\,dm < \infty$.

Theorem 5 (Other Formulation). For a measurable $f : E \to [0, \infty)$,

$$\int_E f\,dm = \sup_{\substack{0 \le \phi \le f \\ \phi \text{ simple}}} \int_E \phi\,dm^{(*)}.$$

Theorem 6 (Equality of Integrals). If f is Riemann integrable on $[a, b]$, then f is measurable and

$$\int_{[a,b]} f\,dm = \int_a^b f(x)\,dx.$$

Theorem 7 (Simple Properties of Lebesgue Integral). Let $f, g : E \to \mathbb{R}$ be measurable whose integral can be defined.

(i) $\int_E (f \pm g)\,dm = \int_E f\,dm \pm \int_E g\,dm$ (if RHS is not of the form $\infty - \infty$).

(ii) For every $c \in \mathbb{R}$, $\int_E cf\,dm = c\int_E f\,dm$ (treat $0 \cdot \infty = 0$).

(iii) If $f \le g$ a.e. on E, then $\int_E f\,dm \le \int_E g\,dm$, equality holds iff $f = g$ a.e..

(iv) $\left|\int_E f\,dm\right| \le \int_E |f|\,dm$.

(v) For disjoint measurable $S, T \subseteq E$, $\int_{S \sqcup T} f\,dm = \int_S f\,dm + \int_T f\,dm$.

(vi) Let $f \ge 0$, then $A \subseteq B \subseteq E \implies \int_A f\,dm \le \int_B f\,dm$.

Theorem 8 (Monotone Convergence, MCT).

- **Increasing Version.** Let $f_1, f_2, \cdots : E \to [0, \infty)$ be measurable functions. Suppose for a.e. $x \in E$, $f_1(x) \le f_2(x) \le \cdots^{(\dagger)}$ and $\lim_{n\to\infty} f_n(x) \in \mathbb{R}$, then

$$\lim_{n\to\infty} \int_E f_n\,dm = \int_E \lim_{n\to\infty} f_n\,dm.$$

- **Decreasing Version.** Let $f_1, f_2, \cdots : E \to \mathbb{R}$ be measurable functions. Suppose for a.e. $x \in E$, $f_1(x) \ge f_2(x) \ge \cdots$ and $\lim_{n\to\infty} f_n(x) \in \mathbb{R}$, then

$$\int_E f_1\,dm < \infty \implies \lim_{n\to\infty} \int_E f_n\,dm = \int_E \lim_{n\to\infty} f_n\,dm.$$

Theorem 9 (Lebesgue Dominated Convergence, LDCT). Let $f_1, f_2, \cdots : E \to \mathbb{R}$ be measurable functions. Suppose that:

(i) f_n converges pointwise for a.e. $x \in E$.

(ii) There is a Lebesgue integrable $g : E \to \mathbb{R}$ s.t. $|f_n| \le g$ a.e. on E, $\forall n \in \mathbb{N}$.

Then $\lim_{n\to\infty} f_n$ measurable on E, moreover,

$$\lim_{n\to\infty} \int_E f_n\,dm = \int_E \lim_{n\to\infty} f_n\,dm.$$

Theorem 10 (Monotone Set). Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue integrable.

(i) $\{A_n\} \nearrow \implies \lim_{n\to\infty} \int_{A_n} f\,dm = \int_{\lim_{n\to\infty} A_n} f\,dm = \int_{\bigcup_{n=1}^\infty A_n} f\,dm$

(ii) $\{A_n\} \searrow \implies \lim_{n\to\infty} \int_{A_n} f\,dm = \int_{\lim_{n\to\infty} A_n} f\,dm = \int_{\bigcap_{n=1}^\infty A_n} f\,dm.$

$^{(*)}$ A proof can be found in page 85 of transparency of this course.

$^{(\dagger)}$ We say that $\{f_n\}$ (or f_n) is **pointwise increasing**. The term **pointwise decreasing** is similarly defined.

Example 1 (Positivity on E, $m(E) > 0$). Let $f : A \to \mathbb{R}$ be measurable functions. Suppose that $E \subseteq A$ is measurable and $f(x) > 0$ for each $x \in E$, show that

$$m(E) > 0 \implies \int_E f\, dm > 0$$

without using Chebyshev's Inequality.

Solution. **Idea.** Since for each $x \in E$ the inequality $f(x) > 0$ holds, we can make use of this inequality to describe the set E.

Assume $m(E) > 0$. Now for each $x \in E$, $f(x) > 0$, so there is an $n \in \mathbb{N}$ such that $f(x) > 1/n$, i.e.,

$$x \in \left\{ t \in E : f(t) > \frac{1}{n} \right\} =: E_n,$$

for some n, therefore $x \in \bigcup_{n=1}^\infty E_n$. The implication says that $E \subseteq \bigcup_{n=1}^\infty E_n$. As by definition $E_n \subseteq E$ for each n, we have $E = \bigcup_{n=1}^\infty E_n$.

By subadditivity of Lebesgue measure,

$$0 < m(E) \le \sum_{n=1}^\infty m(E_n),$$

so there must be an n such that $m(E_n) > 0$. Now we have

$$f > \frac{1}{n} \text{ on } E_n \quad \text{and} \quad f > 0 \text{ on } E \setminus E_n,$$

therefore we have

$$f\chi_{E_n} + f\chi_{E\setminus E_n} > \frac{1}{n}\chi_{E_n} + 0\chi_{E\setminus E_n} \text{ on } E \qquad \iff \qquad f > \frac{1}{n}\chi_{E_n} \text{ on } E.$$

Integration on both sides yields

$$\int_E f(x)\, dm \ge \frac{1}{n} m(E_n) > 0.$$

Remark. Note that we have an important corollary here:

Corollary. *Suppose that $f : A \to [0, \infty)$ is measurable. If $\int_A f\, dm = 0$, then $f = 0$ a.e. on A.*

Proof. Define $E := \{x \in A : f(x) > 0\}$, then we have $f > 0$ on E (by definition) and since $E \subseteq A$ we have

$$0 \le \int_E f\, dm \le \int_A f\, dm = 0 \implies \int_E f\, dm = 0.$$

Therefore by the contrapositive of the statement in Example 1, we have

$$\int_E f\, dm = 0 \implies m(E) = 0,$$

we can conclude $f \le 0$ a.e. on A. Since $f \ge 0$ on A, we have $f = 0$ a.e. on A. ∎

Remark. Note that if we take the corollary for granted (namely, suppose that we can prove this statement in another way), then the statement in Example 1 can be proved in the following way:

Suppose $m(E) > 0$ and $f > 0$ on E, if it happens that $\int_E f\, dm = 0$, then $f = 0$ a.e. on E. Namely, $m(E) = m(E \cap \{x : f(x) = 0\}) > 0$, which implies that $E \cap \{x : f(x) = 0\} \neq \emptyset$, a contradiction to that $f > 0$ on E.

Exercise 1. Suppose $f : A \to \mathbb{R}$ is nonnegative measurable or Lebesgue integrable, then for any $c \in \mathbb{R}$ and measurable $E \subseteq A$ such that $E + c \subseteq A$, show that

$$\int_E f\, dm = \int_{E+c} f(x - c)\, dm(x).$$

Hint: Try to use Theorem 5. First, prove the result is true when f is a simple function; second, we assume $f \ge 0$; third, translate the result to general integrable functions by using the definition: $\int_E f\, dm = \int_E f^+\, dm - \int_E f^-\, dm$.

Example 2. Let $f : \mathbb{R} \to [0,\infty)$ be measurable such that $\int_{\mathbb{R}} \sqrt[3]{f}\, dm, \int_{\mathbb{R}} f^3\, dm < \infty$, prove that
$$\int_{\mathbb{R}} f\, dm < \infty.$$

Solution. **Method 1.** By AM-GM inequality $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$ for $a,b,c,d \geq 0$, we have
$$f \leq \frac{f^3 + \sqrt[3]{f} + \sqrt[3]{f} + \sqrt[3]{f}}{4} = \frac{1}{4} f^3 + \frac{3}{4}\sqrt[3]{f},$$
therefore
$$\int_{\mathbb{R}} f\, dm \leq \frac{1}{4}\int_{\mathbb{R}} f^3\, dm + \frac{3}{4}\int_{\mathbb{R}} \sqrt[3]{f}\, dm < \infty.$$

Method 2. We want to compare f with $\sqrt[3]{f}$ and f^3, but

- $f \leq \sqrt[3]{f}$ only when $f(x) \in [0,1] \iff x \in f^{-1}[0,1]$; and
- $f \leq f^3$ only when $f(x) \in [1,\infty) \iff x \in f^{-1}[1,\infty)$,

it follows that
$$\begin{aligned}\int_{\mathbb{R}} f\, dm &= \int_{f^{-1}[0,1]} f\, dm + \int_{f^{-1}(1,\infty)} f\, dm \\ &\leq \int_{f^{-1}[0,1]} \sqrt[3]{f}\, dm + \int_{f^{-1}(1,\infty)} f^3\, dm \\ &\leq \int_{\mathbb{R}} \sqrt[3]{f}\, dm + \int_{\mathbb{R}} f^3\, dm \\ &< \infty.\end{aligned}$$

Exercise 2. Let $f : \mathbb{R} \to [0,\infty)$ be Lebesgue integrable.

(a) Show that for any $r > 0$, $\displaystyle\int_{[r,\infty)} \frac{f(x)}{x}\, dm < \infty$.

(b) If further $f(0) = 0$ and $f'(0)$ eixsts, then show that $\displaystyle\int_{(0,\infty)} \frac{f(x)}{x}\, dx < \infty$.

Example 3. Let $f : E \to [0,\infty)$ be Lebesgue integrable and $\int_E f\, dm = c \in (0,\infty)$. Prove (2) in the following equalities:
$$\lim_{n\to\infty} \int_E n \ln\left(1 + \left(\frac{f}{n}\right)^{\alpha}\right) dm = \begin{cases} \infty, & \text{if } \alpha \in (0,1), & (1) \\ c, & \text{if } \alpha = 1, & (2) \\ 0, & \text{if } 1 < \alpha < \infty. & (3)\end{cases}$$

Solution. When $\alpha = 1$, we need to prove
$$\lim_{n\to\infty} \int_E \ln\left(1 + \frac{f}{n}\right)^n dm = \int_E f\, dm.$$
Note that for every $y \geq 0$,
$$\left(1 + \frac{y}{n}\right)^n \nearrow e^y,$$
we prove this in the remark below. Therefore we have
$$\ln\left(1 + \frac{f(x)}{n}\right)^n \nearrow f(x)$$
pointwise on E, so we are done by using MCT.

Remark. **Method 1.** This method is elementary. We fix a $y \in [0,\infty)$, by AM-GM inequality,
$$\left(\left(1 + \frac{y}{n}\right)^n \cdot 1\right)^{\frac{1}{n+1}} \leq \frac{n(1 + \frac{y}{n}) + 1}{n+1} \iff \left(1 + \frac{y}{n}\right)^n \leq \left(1 + \frac{y}{n+1}\right)^{n+1}.$$

Method 2. This method is nonelementary, but very routine. Let's define $f(x) = (1 + \frac{y}{x})^x$ and show that f is increasing on $(0,\infty)$. For this, note that $f'(x) > 0 \iff \ln\frac{x+y}{x} > \frac{y}{x+y} = 1 - \frac{x}{x+y}$. Put $u = \frac{x+y}{x}$, the above inequality becomes $\ln u > 1 - 1/u$, where $u > 1$.

Exercise 3. Prove (1) and (3) in the equalities given in Example 3.

Exercise 4 (Corollaries of MCT). Prove the important corollaries of MCT:

Interchangeability of Summation and Integration Let $f_1, f_2, \cdots : E \to [0,\infty)$ be measurable functions, if $\sum_{n=1}^{\infty} f_n(x) < \infty$ for a.e. $x \in E$, then
$$\int_E \sum_{n=1}^{\infty} f_n(x)\, dm = \sum_{n=1}^{\infty} \int_E f_n(x)\, dm.$$

Countable Additivity as a Set Function Let $f : E \to [0,\infty)$ be measurable. Suppose $E_1, E_2, \cdots \subseteq E$ is a sequence of pairwise disjoint measurable sets, then
$$\int_{\bigsqcup_{n=1}^{\infty} E_n} f\, dm = \sum_{n=1}^{\infty} \int_{E_n} f\, dm.$$

Example 4. Let $a > 0$, show that

$$\lim_{n\to\infty} \int_a^\infty \frac{n^2 x e^{-n^2x^2}}{1+x^2}\,dx = 0.$$

Solution.

$$\begin{aligned}\int_a^\infty \frac{n^2 x e^{-n^2x^2}}{1+x^2}\,dx &= \int_{na}^\infty \frac{xe^{-x^2}}{1+x^2/n^2}\,dx \\ &\overset{\text{MCT}}{=} \int_{[na,\infty)} \frac{xe^{-x^2}}{1+x^2/n^2}\,dm \\ &= \int_{[0,\infty)} \chi_{[na,\infty)} \frac{xe^{-x^2}}{1+x^2/n^2}\,dm.\end{aligned}$$

It is obvious that the integrant converges to 0 pointwise, so it is enough to check the interchangeability of integral sign and limit.

As the "pointwise monotonicity" of the integrand is not very clear, we try to "dominate" the integrand(s) (i.e., find g s.t. $|\cdot_n| \le g$) in order to apply LDCT. First,

$$\left|\chi_{[na,\infty)} \frac{xe^{-x^2}}{1+x^2/n^2}\right| \le xe^{-x^2},$$

since $g(x) := xe^{-x^2}$ is Lebesgue integrable on $[0,\infty)$, so by LDCT we have

$$\begin{aligned}\lim_{n\to\infty}\int_a^\infty \frac{n^2 x e^{-n^2x^2}}{1+x^2}\,dx &= \lim_{n\to\infty}\int_{[0,\infty)} \chi_{[na,\infty)} \frac{xe^{-x^2}}{1+x^2/n^2}\,dm \\ &= \int_{[0,\infty)} \lim_{n\to\infty} \chi_{[na,\infty)} \frac{xe^{-x^2}}{1+x^2/n^2}\,dm \\ &= \int_{[0,\infty)} 0\,dm = 0.\end{aligned}$$

Exercise 5 (Subadditivity). Let $f : E \to [0,\infty)$ be measurable. By using the Exercise 4, show that the set function $A \mapsto \int_A f\,dm$ is **subadditive**:

$$A_i\text{'s measurable} \implies \textstyle\int_{\bigcup_{n=1}^\infty A_n} f\,dm \le \sum_{n=1}^\infty \int_{A_n} f\,dm.$$

Exercise 6 (Countable Additivity). Let $f : E \to \mathbb{R}$ be Lebesgue integrable. By using LDCT, show that if A_k's are pairwise disjoint measurable set, then

$$\textstyle\int_{\bigsqcup_{k=1}^\infty A_k} f\,dm = \sum_{k=1}^\infty \int_{A_k} f\,dm.$$

Example 5. Let E be measurable and $f : E \to \mathbb{R}$ Lebesgue integrable on E. Define $E_k = \{x \in E : |f(x)| < \frac{1}{k}\}$, show that

$$\lim_{k\to\infty} \int_{E_k} |f|\,dm = 0.$$

Solution. We observe that

$$x \in E_k \implies x \in E_{k-1},$$

therefore $E_k \subseteq E_{k-1}$, i.e., $\{E_k\}$ is descending, we hope to use integration version of MST.

Indeed, since f is Lebesgue integrable, we have by using the integration version of MST,

$$\lim_{k\to\infty}\int_{E_k} |f|\,dm = \int_{\bigcap_{k=1}^\infty E_k} |f|\,dm.$$

Since $f = 0$ on $\bigcap_{k=1}^\infty E_k$, so

$$\lim_{k\to\infty}\int_{E_k} |f|\,dm = 0.$$

Exercise 7 (2012 Final). Let $f : [0,1] \to \mathbb{R}$ be increasing, prove that for every $E \subseteq [0,1]$, $m(E) = t$, we have $\displaystyle\int_{[0,t]} f\,dm \le \int_E f\,dm$.

Hint: Use the previous exercises. We have two methods to do this.

Exercise 8. Let $f : [a,b] \to (0,\infty)$ be Lebesgue integrable and $\{E_n\}$ a collection of Lebesgue measurable subsets of $[a,b]$. Show that if

$$\lim_{n\to\infty}\int_{E_n} f\,dm = 0,$$

then $\lim_{n\to\infty} m(E_n) = 0$.

Hint: Use the previous exercises.