Math2033 Mathematical Analysis (Spring 2013-2014) **Tutorial Note 10**

Riemann Integral (Part III): Improper Integral

We need to know

- how to judge whether a function is improper integrable by using suitable tests;

Key definitions and results

Definition 1 (Locally Integrability). A function $f(x)$ defined on an interval I (bounded or unbounded) is said to be **locally integrable** if it is integrable on any closed subinterval of I.

Definition 2 (Improper Integrals). Let $f(x)$ be locally integrable on its domain. We define **improper integrals** in the following cases:

Case 1. If $f(x)$ is *not defined* **just at *a*** / **just at *b*** / **just at $c \in (a,b)$** / **just at both *a* and *b*** of a bounded interval $[a,b]$, then we define:

$$\int_a^b f(x)\,dx = \begin{cases} \lim\limits_{\epsilon\to0^+}\int_{a+\epsilon}^b f(x)\,dx, & \text{if } f \text{ is not defined } \textbf{at } \boldsymbol{a}, \\ \lim\limits_{\epsilon\to0^+}\int_a^{b-\epsilon} f(x)\,dx, & \text{if } f \text{ is not defined } \textbf{at } \boldsymbol{b}, \\ \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, & \text{if } f \text{ is not defined at } \boldsymbol{c\in(a,b)} \\ \lim\limits_{\substack{\epsilon_1\to0^+\\ \epsilon_2\to0^+}}\int_{a+\epsilon_1}^{b-\epsilon_2} f(x)\,dx, & \text{if } f \text{ is not defined } \textbf{at } \boldsymbol{a,b}. \end{cases}$$

Case 2. If $f(x)$ is defined on an unbounded interval, then we define the corresponding improper integrals in a natural way:

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx, \quad \int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

and

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{\substack{a\to-\infty\\ b\to\infty}}\int_a^b f(x)\,dx.$$

Remark. In other words, we do integration locally and take limit, that's all. The spirit is: Riemann integral is only defined on a closed and bounded interval, we extend this notion via taking limit.

Remark. For **nonnegative** locally integrable function $f(x)$ on I we have $f(x)$ is improper integrable if and only if $\int_I f(x)\,dx < \infty$.

Definition 3 (Principal Value). Let $f(x)$ be locally integrable on I.

(a) If $I = [a,c)\cup(c,b]$, then the principal value of $\int_a^b f(x)\,dx$ is defined by

$$\text{P.V.}\int_a^b f(x)\,dx = \lim_{\epsilon\to0^+}\left(\int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^b f(x)\,dx\right).$$

(b) If $I = \mathbb{R}$, then the principal value of $\int_{-\infty}^\infty f(x)\,dx$ is defined by

$$\text{P.V.}\int_{-\infty}^\infty f(x)\,dx = \lim_{c\to\infty}\int_{-c}^c f(x)\,dx.$$

Theorem 4 (Comparison Test). Suppose $f(x), g(x)$ are locally integrable on a bounded or unbounded interval I and $0 \le f(x) \le g(x)$ on I, then

$$g(x) \text{ is improper integrable} \implies f(x) \text{ is improper integrable.}$$

Remark. Taking contrapositive, we have

$$f(x) \text{ is not improper integrable} \implies g(x) \text{ is not improper integrable.}$$

Theorem 5 (Limit Comparison Test). Suppose $f(x), g(x) > 0$ are locally integrable on $(a,b]$.

(a) If $\lim\limits_{x\to a^+}\frac{f(x)}{g(x)} = L$ for some positive number L, then either $f(x), g(x)$ are *both improper integrable* or *both not improper integrable*.

(b) If $\lim\limits_{x\to a^+}\frac{f(x)}{g(x)} = 0$, then $g(x)$ is improper integrable $\implies$ $f(x)$ is improper integrable.

Theorem 6 (Absolute Convergence Test). Let $f(x)$ be locally integrable on I. If $|f(x)|$ is improper integrable on I, then so is $f(x)$.

Example 1. Discuss the existence of the following improper integrals:

(a) $\displaystyle\int_0^1 \frac{e^{x^{2014}} \ln(1+\sin(x^2))}{x^{5/2}}\, dx$

(b) $\displaystyle\int_1^\infty \frac{1}{x^{1+\frac{4}{x}}\sqrt{x^2+x^{1/2}+\sin x+1}}\, dx$

(c) $\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{1+x}\, dx$

Sol

Example 2.

(a) When $p > 1$, does the improper integral $\displaystyle\int_1^\infty \frac{\sin x}{x^p}\,dx$ exist?

(b) Show that $\displaystyle\int_0^\infty \left|\frac{\sin x}{x}\right| dx = \infty$ and $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ exist.

Sol

Example 3. Study the convergence of $\displaystyle\int_{-\infty}^\infty \sin x\,dx$, what is its principal value? How about the convergence of $\displaystyle\int_{-\infty}^\infty \sin(x^2)\,dx$?

Sol

Exercises

1. Find a locally integrable function $f(x)$ on $[0,\infty)$ such that it is improper integrable but $\lim_{x\to\infty} f(x)$ does not exist.

2. Let $f:[a,\infty)\to\mathbb{R}$ be locally and improper integrable, i.e., $\int_a^\infty f(x)\,dx$ converges. Show that if $f(x)$ is uniformly continuous, then $\lim_{x\to\infty} f(x)=0$.

3. (2007 Spring Final) Does the improper integral $\int_{-1}^1 \frac{1}{x\cos x}\,dx$ exist?

4. Determine the convergence of the following improper integrals:

(a) $\int_0^{1/4} \frac{1}{\sqrt{x}(1-x)}\,dx$ (b) $\int_0^1 \frac{1}{\sqrt{x}(1-x)}\,dx$ (c) $\int_0^1 \frac{\sin\frac{1}{x}}{x^p}\,dx$, $p>0$

(d) $\int_1^\infty \frac{\sin x}{x^p+\sin x}\,dx$, $p>0$ (e) $\int_0^\infty \frac{x}{1+x^p\cos^2 x}\,dx$, $p>4$

5. Prove that the improper integral $\int_0^\pi \ln(\sin x)\,dx$ exists and compute it.

6. For any $a\in(0,1]$, show that the improper integral $\int_0^1 \left(\left[\frac{a}{x}\right]-a\left[\frac{1}{x}\right]\right)dx$ exists, moreover, it is equal to $a\ln a$.

7. Let $f(x)$ be nonegative and locally integrable on $[0,\infty)$. Suppose that $f(x)$ is improper integrable, i.e., $\int_0^\infty f(x)\,dx<\infty$.

(a) If f is decreasing, show that $\lim_{x\to\infty} xf(x)=0$.

(b) If f merely improper integrable, show that $\lim_{n\to\infty}\frac{1}{n}\int_0^n xf(x)\,dx=0$.

8. Show that

$$\int_{-\pi}^{\pi} \underbrace{\frac{\sin\left((N+\frac{1}{2})x\right)}{\sin\frac{x}{2}}}_{:=D_N(x)}\frac{dx}{2\pi}=1.$$

Here $D_N(x)$ is called **Dirichlet kernel**. It naturally arises in the study of pointwise convergence of Fourier series (deeper study will be left to Math4052).

9. We still use $D_N(x)$ to denote Dirichlet kernel defined in Exercise 8. We have shown that the improper integral $\int_0^\infty \frac{\sin x}{x}\,dx$ exists in part (b) of Example 2. In this exercise we try to show that $\int_0^\infty \frac{\sin x}{x}\,dx=\frac{\pi}{2}$ as a byproduct of machinery developed for studying Fourier series.

(a) Show that $\frac{1}{\sin\frac{x}{2}}-\frac{2}{x}$ is bounded on $(0,\pi]$, then prove that

$$\lim_{N\to\infty}\int_0^\pi\left(\frac{1}{\sin\frac{x}{2}}-\frac{2}{x}\right)\sin(N+\tfrac{1}{2})x\,dx=0.$$

Hint: Use the Riemann-Lebesgue Lemma in Example 4 of Tutorial Note 9. Oh yes! You need to extend $\frac{1}{\sin\frac{x}{2}}-\frac{2}{x}$ continuously at $x=0$.

(b) By using Exercise 8 and part (a) above, show that $\int_0^\infty \frac{\sin x}{x}\,dx=\frac{\pi}{2}$.

Remark. This improper integral is a textbook example to demonstrate techniques in computing improper integrals. Students will revisit this integral in Math3033 (using Lebesgue Dominated Convergence Theorem) and Math 4023 (using Residue Calculus).