Riemann Integral (Part II): Miscellaneous

We need to know

- how to judge a set is of measure zero;
- apart from showing continuous functions are Riemann integrable, how useful is uniform continuity?

Key definitions and results

Definition 1 (Uniform Continuity). A function $f: S \to \mathbb{R}$ is said to be **uniformly continuous** if there holds

$$\forall \epsilon > 0, \exists \delta > 0 \quad \text{s.t.} \quad |x-y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Definition 2 (Measure Zero, a.e. Property).

- A set $S \subseteq \mathbb{R}$ is of **measure zero** if for every $\epsilon > 0$, there are open intervals $I_1, I_2, \ldots$ such that $S \subseteq \bigcup_{i=1}^{\infty} I_i$ and $\sum_{i=1}^{\infty} |I_i| < \epsilon$.
- We say a property $P = P(x)$ holds **a.e.** if the set $\{x : P(x) \text{ does not hold}\}$ has measure zero.

Remark. It is immediate from the definition that a subset of a set of measure zero is of measure zero; also, a union of countably many sets of measure zero is again of measure zero.

Theorem 3. Let $f(x)$ be a continuous function on $[a,b]$, then $f(x)$ is uniformly continuous there.

Theorem 4. If $f: [a,b] \to \mathbb{R}$ is Riemann integrable and $c \in [a,b]$, then $F(x) := \int_c^x f(t)\,dt$ is uniformly continuous on $[a,b]$.

Theorem 5 (Fundamental Theorem of Calculus). Let $c, x_0 \in [a,b]$.

(a) If $f(x)$ is Riemann integrable on $[a,b]$, continuous at x_0 and $F(x) = \int_c^x f(t)\,dt$, then $F'(x_0) = f(x_0)$.

(b) If $G(x)$ is differentiable on $[a,b]$ with $G'(x) = g(x)$ Riemann integrable on $[a,b]$, then $\int_a^b g(x)\,dx = G(b) - G(a)$.

Theorem 6 (Composition). If $f(x)$ is Riemann integrable on $[a,b]$ and $\phi(x)$ is continuous on $f([a,b])$, then $\phi(f(x))$ is Riemann integrable on $[a,b]$.

Theorem 7 (Lebesgue Criterion). Let $f(x)$ be bounded on $[a,b]$, then $f(x)$ is Riemann integrable on $[a,b]$ if and only if

$$S_f := \{x \in [a,b] : f \text{ is discontinuous at } x\}$$

is of measure zero.

Theorem 8 (Integration by Parts). If $f(x), g(x)$ are differentiable on $[a,b]$ and $f'(x), g'(x)$ are Riemann integrable on $[a,b]$, then

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx.$$

Theorem 9 (Change of Variable Formula). If $\phi(x)$ is differentiable, $\phi'(x)$ is integrable on $[a,b]$ and $f(x)$ is continuous on $\phi([a,b])$, then

$$\int_{\phi(a)}^{\phi(b)} f(t)\,dt = \int_a^b f(\phi(x))\phi'(x)\,dx.$$

Remark. The continuity of $\phi(x)$ in Theorem 6 cannot be replaced by Riemann integrability. For example, the **Thomae's function** $f: [0,1] \to \mathbb{R}$ given by $f(0) = 0$ and for $x \in (0,1]$,

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q} \ \ p, q \text{ coprime} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is Riemann integrable. The function $\phi: [0,1] \to \{0,1\}$ given by $\phi(0) = 0$ and $\phi(x) = 1$ for $x > 0$ is also Riemann integrable, but their composition

$$\phi \circ f = \begin{cases} 1 & x \in [0,1] \cap \mathbb{Q} \\ 0 & x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

is not Riemann integrable.

Conclusion: Generally the composition of two Riemann integrable functions is not Riemann integrable.

Example 1.

(a) If $f(x)$ has bounded derivative on an (bounded or unbounded) open interval I, show that $f(x)$ is uniformly continuous on I.

(b) Show that for any $p \in (0,1)$, $h(x) := x^p$ is uniformly continuous on $[0,\infty)$.

(c) Show that $f(x) = x^2$ is **not** uniformly continuous on $[0,\infty)$.

Sol **(a)** As f' is bounded on I, there is $M > 0$ such that $|f'| \le M$ on I. Now for any $x, y \in I$, Mean-Value Theorem says that there is c between x, y,

$$|f(x) - f(y)| = |f'(c)(x-y)| \le M|x-y|.$$

Now for every $\epsilon > 0$, we may choose $\delta = \epsilon/M$, then

$$|x-y| < \delta \implies |f(x)-f(y)| < M\delta = \epsilon.$$

(b) We know that for every $x, y \ge 0$,

$$|x^p - y^p| \le |x-y|^p,$$

and thus for every $\epsilon > 0$, the choice $\delta = \epsilon^{1/p}$ will do.

(c) We try to show

$$\begin{aligned} & \text{nonuniform continuity of } f \\ \iff & \sim(\forall\epsilon > 0, \exists\delta > 0, |x-y| < \delta \implies |f(x)-f(y)| < \epsilon) \\ \iff & \exists\epsilon > 0, \forall\delta > 0, \exists x, y \in [0,\infty), |x-y| < \delta \quad \text{and} \quad |f(x)-f(y)| \ge \epsilon \end{aligned}$$

for any sequence $\{\delta_n\}$ s.t. $\delta_n \to 0$, the statement above is equivalent to

$$\iff \exists\epsilon > 0, \forall n, \exists x_n, y_n \in [0,\infty), |x_n - y_n| < \delta_n \quad \text{and} \quad |f(x_n) - f(y_n)| \ge \epsilon.$$

Therefore, the following are equivalent:

- $f : A \to B$ is not uniformly continuous.
- There is a sequence of arbitrarily close pairs in the domain whose images can't be arbitrarily close.

In particular, let's choose $x_n = \sqrt{n+1}$ and $y_n = \sqrt{n}$, then $|x_n - y_n| = \frac{1}{\sqrt{n+1}+\sqrt{n}}$, so x_n, y_n's are arbitrarily close pairs, but

$$|f(x_n) - f(y_n)| = |(n+1) - n| = 1 \ge 1$$

is bounded below by a positive constant. ∎

Remark. To consolidate understanding, please try to show that $f(x) = \sin(x^2)$ is **not** uniformly continuous on $\mathbb{R}$.

In other words, try to construct a sequence of pairs x_n, y_n such that $x_n - y_n \to 0$ but $|f(x_n) - f(y_n)|$ can't be arbitrarily small.

Example 2 (2002 Final). Let $f, g : [0,2] \to \mathbb{R}$ be Riemann integrable. Prove that $h : [0,2] \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} \max\{f(x), g(x)\} & \text{if } x \in [0,1] \\ \min\{f(x), g(x)\} & \text{if } x \in (1,2] \end{cases}$$

is also Riemann integrable on $[0,2]$.

Sol We try to show

$$S_h = \{x \in [0,2] : h \text{ not cont. at } x\}$$

has measure zero.

Due to the way we define h, let's decompose S_h into

$$S_h = (S_h \cap [0,1)) \cup (S_h \cap (1,2]) \cup (\boldsymbol{S_h \cap \{1\}})$$

Since $h = \max\{f, g\}$ on $[0,1)$, we have

$$S_h \cap [0,1) = S_{\max\{f,g\}} \cap [0,1).$$

Similarly,

$$S_h \cap (1,2] = S_{\min\{f,g\}} \cap (1,2].$$

Therefore we conclude

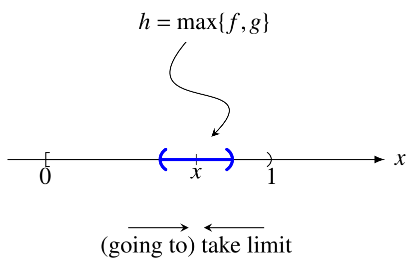
$$S_h \subseteq S_{\max\{f,g\}} \cup S_{\min\{f,g\}} \cup \{1\}.$$

Remark. We need to decompose S_h into 3 pieces rather than 2 pieces: $[0,1]$ and $(1,2]$. The reason is $\max\{f,g\}$ can be continuous at 1 with h discontinuous at 1. An example is $f \equiv 2$ and $g \equiv 1$, we have $h = \max\{f,g\} = 2$ on $[0,1]$ and $h = \min\{f,g\} = 1$ on $(1,2]$.

The trouble arises as h is **"redefined"** on the right of 1, continuity of $\max\{f,g\}$ at 1 **does not imply**

$$\lim_{x \to 1^-} \underbrace{h(x)}_{=\max\{f(x),g(x)\}} = \lim_{x \to 1^+} \underbrace{h(x)}_{=\min\{f(x),g(x)\}}.$$

Moreover, for every $x \in [0,1)$ (similarly for $x \in (1,2]$), $h = \max\{f,g\}$ completely on a sufficiently small neighborhood of x (including both LHS and RHS of x), therefore $S_h \cap [0,1) = S_{\max\{f,g\}} \cap [0,1)$ must be true:



The uncertainty $S_h \cap \{1\}$ courses no trouble since it has measure zero.

Now it is enough to show $S_{\max\{f,g\}}$ and $S_{\min\{f,g\}}$ have measure zero. This can be done in two ways:

Method 1. By the formula

$$\max\{x, y\} = \frac{1}{2}(x + y + |x - y|) \quad \text{and} \quad \min\{x, y\} = \frac{1}{2}(x + y - |x - y|)$$

we see that both $\max\{f,g\}$ and $\min\{f,g\}$ are Riemann integrable, hence both $S_{\max\{f,g\}}, S_{\min\{f,g\}}$ have measure zero by Lebesgue Theorem. ∎

Method 2. Since f cont. at x and g cont. at $x \implies \max\{f,g\}$ cont. at x, by contrapositive, we have

$$S_{\max\{f,g\}} \subseteq S_f \cup S_g.$$

Since a union of two measure zero sets are of measure zero, $S_f \cup S_g$ has measure zero. $S_{\max\{f,g\}}$ being a **subset** of measure zero set is also of measure zero.

Similarly, since $S_{\min\{f,g\}} \subseteq S_f \cup S_g$, so $S_{\min\{f,g\}}$ has measure zero. ∎

Example 3. Let $f(x)$ be Riemann integrable on $[a,b]$. Suppose that

$$\int_a^b f(x)\,dx > 0,$$

show that there is an $\eta > 0$ and a closed subinterval I such that $f(x) > \eta$ on I.

Sol We prove by contradiction.

Suppose on the contrary for every $\eta > 0$, for every subinterval I we can find an $x \in I$, $f(x) \leq \eta$.

Let $\epsilon > 0$ (to be taken small) be fixed and consider the Riemann integral of $f(x)$.

Let $P = \{x_0, x_1, \ldots, x_n\}$ 🔪 $[a,b]$. For each subinterval $[x_{i-1}, x_i]$, there is $x_i^* \in [x_{i-1}, x_i]$ s.t. $f(x_i^*) \leq \epsilon$, and therefore

$$L(f,P) \leq S(f,P) = \sum_{i=1}^{n} f(x_i^*)\Delta x_i \leq \epsilon \sum_{i=1}^{n}(x_i - x_{i=1}) = \epsilon(b-a).$$

Now we take $\|P\| \to 0$ such that $L(f,P) \to \int_a^b f(x)\,dx$, we have

$$\int_a^b f(x)\,dx \leq \epsilon(b-a).$$

Since $\epsilon > 0$ is arbitrary, we can take $\epsilon \to 0^+$ to conclude $\int_a^b f(x)\,dx \leq 0$, a contradiction to that $\int_a^b f(x)\,dx > 0$. ∎

Example 4 (Riemann-Lebesgue Lemma). Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{w\to\infty} \int_a^b f(x)\sin(wx)\,dx = 0.$$

Remark. We generalize this result in Exercise 8 and 9, this is an important result in the study of pointwise convergence of Fourier series and will be used in Math4052.

Sol First of all, let's approximate $f(x)$ closely by a step function. A step function is a function that is piecewise constant on finitely many intervals.

Since $f(x)$ is continuous on $[a,b]$, it is uniformly continuous there.

Let $\epsilon > 0$ be given, there is $\delta > 0$ such that for every $x, y \in [a,b]$,

$$|x-y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Let's take $a = x_0 < x_1 < \cdots < x_n = b$ such that $\max_{1\leq i\leq n}(x_i - x_{i-1}) < \delta$. We construct a step functions as follows:

$$\phi(x) = f\Big(\underbrace{\frac{x_i + x_{i-1}}{2}}_{:=m_i}\Big) \quad \text{for } x \in \underbrace{[x_{i-1}, x_i)}_{:=J_i},\ i < n$$

and

$$\phi(x) = f\Big(\underbrace{\frac{x_n + x_{n-1}}{2}}_{:=m_n}\Big) \quad \text{for } x \in \underbrace{[x_{n-1}, x_n]}_{:=J_n}.$$

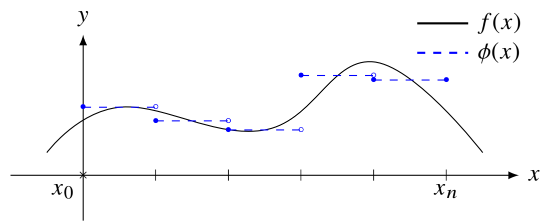
This step function satisfies

$$|f(x) - \phi(x)| < \epsilon. \tag{$*$}$$

To explain $(*)$, let $x \in [a,b]$, then there is an i, $x \in J_i$, and hence

$$|x - m_i| < \delta \implies |f(x) - \phi(x)| = |f(x) - f(m_i)| < \epsilon.$$

$(*)$ is pictorially clear for those partition that is refined enough:



Note that we don't know how to compute $\int_a^b f(x)\sin(wx)\,dx$ but we do know how to compute $\int_a^b \phi(x)\sin(wx)\,dx$—that's the key point of the proof.

Now for the **fixed** $\phi(x)$ constructed above,

$$\begin{aligned}\left|\int_a^b f(x)\sin(wx)\,dx\right| &\le \left|\int_a^b (f(x)-\phi(x))\sin(wx)\,dx\right| + \left|\int_a^b \phi(x)\sin(wx)\,dx\right| \\ &\le \int_a^b |f(x)-\phi(x)|\,dx + \left|\sum_{i=1}^n \int_{x_{i-1}}^{x_i} \phi(x)\sin(wx)\,dx\right| \qquad (\star) \\ &< (b-a)\epsilon + \left|\sum_{i=1}^n \frac{f(m_i)(\cos wx_{i-1}-\cos wx_i)}{w}\right| \\ &\le (b-a)\epsilon + \sum_{i=1}^n \frac{|f(m_i)|\cdot 2}{|w|} \\ &= (b-a)\epsilon + \frac{2\sum_{i=1}^n |f(m_i)|}{|w|}. \qquad (**)\end{aligned}$$

Now we can find a $b>0$ such that

$$w > b \implies \frac{1}{|w|} < \frac{\epsilon}{2\sum_{i=1}^n |f(m_i)|+1},$$

it follows that from $(**)$,

$$w > b \implies \left|\int_a^b f(x)\sin(wx)\,dx\right| < (b-a+1)\epsilon.$$

Thus $\lim_{w\to\infty}\int_a^b f(x)\sin(wx)\,dx = 0$ by the definition of limit. ∎

Remark. In Example 4 the result $\lim_{w\to\infty}\int_a^b f(x)\cos(wx)\,dx = 0$ still holds with the same proof. More generally, the continuity of $f(x)$ can be replaced by Riemann integrability. As we have seen the key step in our solution is to show a function f has the following property:

> ***For a given $\epsilon > 0$, there is a step function ϕ such that $\int_a^b |f(x)-\phi(x)|\,dx < \epsilon$.***

After that we can continue from $(\star)$ to repeat the remaining steps. Riemann integrable functions also have this property:

Theorem. *If $f(x)$ is Riemann integrable on $[a,b]$, then for every $\epsilon>0$, there is a step function $\phi(x)$ on $[a,b]$ such that*

$$\int_a^b |f(x)-\phi(x)|\,dx < \epsilon.$$

Proof. This is very immediate from definition if we can realize that every $L(f, P := \{x_0,\dots,x_n\}) = \sum_{i=1}^n \inf_{[x_{i-1},x_i]} f\Delta x_i$ is actually the Riemann integral of the step function:

$$\varphi = \inf_{[x_{i-1},x_i]} f \quad \text{on } [x_{i-1},x_i), i<n$$

and

$$\varphi = \inf_{[x_{n-1},x_n]} f \quad \text{on } [x_{n-1},x_n].$$

Suppose that P is chosen at the beginning such that $\int_a^b f(x)\,dx - L(f,P) < \epsilon$, then since $L(f,P) = \int_a^b \varphi(x)\,dx$ for φ defined above, we have

$$\int_a^b \underbrace{(f(x)-\varphi(x))}_{\ge 0}\,dx < \epsilon. \qquad ∎$$

With a slight modification on the step function in the proof above, one has:

Corollary. *If $f(x)$ is Riemann integrable on $[a,b]$, then for every $\epsilon>0$, there is a continuous function $g(x)$ on $[a,b]$ such that*

$$\int_a^b |f(x)-g(x)|\,dx < \epsilon.$$

Moreover, such $g(x)$ can be chosen such that $|g| \le \sup_{[a,b]}|f|$ on $[a,b]$.

Finally, everything seems to have nothing to do with Lebesgue. The Riemann-Lebesgue Lemma actually holds for Lebesgue integrable functions as well:

Theorem. *For every $f(x)$ Lebesgue integrable on $[a,b]$,*

$$\lim_{w\to\infty}\int_{[a,b]} f(x)\sin(wx)\,dm(x) = 0.$$

The proof is more subtle, one can still use step functions technique, and in addition we need the **outer regularity** of Lebesgue measure m and we also need L^1 approximation technique using **simple functions**, for detail one can refer to lecture notes of Math3043.

Exercises

1. (a) Show that $f(x) = \sin(x^2)$ is not uniformly continuous on $\mathbb{R}$. How does the graph of $f(x)$ look?

(b) Let $f(x)$ be continuous on $[a,\infty)$, show that if $\lim_{x\to\infty} f(x) = a \in \mathbb{R}$, then $f(x)$ is uniformly continuous on $[a,\infty)$.

2. Let $f(x)$ be continuous on $[-1,1]$. Suppose that $f(x)$ satisfies

$$\int_{-1}^{1} f(x)g(x)\,dx = 0$$

for every **even** integrable function $g(x)$ on $[-1,1]$. Prove that $f(x)$ must be odd.

3. Suppose $f(x)$ is differentiable on $[0,1]$ with $f'(x)$ continuous on $[0,1]$. Show that

$$\sup_{x\in[0,1]} |f(x)| \le \int_0^1 |f(x)|\,dx + \int_0^1 |f'(x)|\,dx.$$

Hint: Recall Extreme Value Theorem for continuous functions on $[0,1]$.

4. Suppose $f(x)$ is continuous on $[a,b]$ with $f(x) > 0$ on (a,b). Suppose also that $g(x)$ is Riemann integrable on $[a,b]$, prove that

$$\lim_{n\to\infty} \int_a^b g(x)\sqrt[n]{f(x)}\,dx = \int_a^b g(x)\,dx.$$

5. (Uniform Continuity on $\mathbb{R}^2$) Show that if $f : [a,b]\times[c,d] \to \mathbb{R}$ is continuous, then $f(x,y)$ is uniformly continuous on $[a,b]\times[c,d]$ in the sense that

$$\forall \epsilon > 0, \exists \delta > 0 \quad \text{s.t.} \quad \|\vec{u}-\vec{v}\| < \delta \implies |f(\vec{u}) - f(\vec{v})| < \epsilon.$$

Hint: You may repeat the proof of its real line analogue; recall that any bounded sequence in $\mathbb{R}^2$ has a convergent subsequence, see Exercise 8 of Tutorial Note 4.

6. (Generalized Fundamental Theorem of Calculus) Let $f(x,t)$ be such that both $f(x,t)$ and $\frac{\partial f}{\partial t}(x,t)$ are continuous on $[a,b]\times[c,d]$ (in the multivariable calculus sense). Let $\phi : (c,d) \to (a,b)$ be differentiable. Show that for every $t \in (c,d)$,

$$\frac{d}{dt}\left(\int_a^{\phi(t)} f(x,t)\,dx\right) = \int_a^{\phi(t)} \frac{\partial f}{\partial t}(x,t)\,dx + f(\phi(t),t)\phi'(t).$$

The following illustrate the step functions technique as in Example 4.

7. (Approximation by Continuous Functions) Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable. Show that for every $\epsilon > 0$, there is a continuous function $g : [a,b] \to \mathbb{R}$ with $|g(x)| \le \sup_{t\in[a,b]} |f(t)|$ on $[a,b]$ such that

$$\int_a^b |f(x) - g(x)|\,dx < \epsilon.$$

Hint: Approximate the integral $\int_a^b f\,dx$ by the definition of Riemann lower sum, this sum can be viewed as an integral of a step function $\boldsymbol{s(x)}$. Modify this step function in a *linear* way to make it become a continuous function $\boldsymbol{g(x)}$. Justify that your modification g satisfies $\int_a^b |g(x) - s(x)|\,dx < \varepsilon$.

8. (Full Version of Riemann-Lebesgue Lemma) By using Exercise 7, generalize Example 4 by requiring now $f(x)$ be merely Riemann integrable on $[a,b]$.

9. (Generalized Riemann-Lebesgue Lemma) Let $f(x)$ be a T-periodic function on $\mathbb{R}$ that is integrable on $[0,T]$, and $g(x)$ integrable on $[a,b]$. Prove that

$$\lim_{w\to\infty} \int_a^b f(wx)g(x)\,dx = \frac{1}{T}\int_0^T f(x)\,dx \int_a^b g(x)\,dx.$$