Lebesgue Measurable Functions

Key Definitions and Results

Definition 1. Let A be a measurable set, we say that $f : A \to \mathbb{R}$ is a **(Lebesgue) measurable function** if $f^{-1}(a,b)$ is a measurable set for every $a,b \in \mathbb{R}$.

Remark. Implicit to saying $f : A \to \mathbb{R}$ is measurable is the measurability of A because $A = f^{-1}(-\infty,\infty)$, see Theorem 3 for detail.

Definition 2. Let $E \subseteq \mathbb{R}$ be measurable and let $P(x)$ be a property related to points $x \in \mathbb{R}$. We say that $P(x)$ holds **almost everywhere (abbr. a.e.) on E**(*) if

$$m\{x \in E : P(x) \text{ does not hold}\} = 0.$$

Theorem 3. The following are equivalent:

(i) $f : A \to \mathbb{R}$ is measurable.

(ii) $f^{-1}[a,b)$ is a measurable set for every $a,b \in \mathbb{R}$, $a < b$.

(iii) $f^{-1}[a,b]$ is a measurable set for every $a,b \in \mathbb{R}$, $a < b$.

(iv) $f^{-1}(a,b]$ is a measurable set for every $a,b \in \mathbb{R}$, $a < b$.

(v) $f^{-1}(-\infty,b]$ is a measurable set for every $b \in \mathbb{R}$.

(vi) $f^{-1}(-\infty,b)$ is a measurable set for every $b \in \mathbb{R}$.

(vii) $f^{-1}[a,\infty)$ is a measurable set for every $a \in \mathbb{R}$.

(viii) $f^{-1}(a,\infty)$ is a measurable set for every $a \in \mathbb{R}$.

Theorem 4 (Topological Continuity). Let $A,B \subseteq \mathbb{R}$, the following are equivalent:

(i) $f : A \to B$ is continuous.

(ii) For every open set U, $f^{-1}(U) = V \cap A$, for some V open.

(iii) For every bounded open interval (a,b), $f^{-1}(a,b) = U \cap A$, for some U open.

(*) Or that $P(x)$ holds for **almost every (abbr. a.e.) $x \in E$**.

Theorem 5 (Summary on Properties of Measurable Functions).

(i) Let A,B be measurable. If $g : B \to \mathbb{R}$ is continuous and $f : A \to B$ is measurable, then $g \circ f : A \to \mathbb{R}$ is also measurable.

(ii) If $f_1, f_2 : A \to \mathbb{R}$ are measurable, then $f_1 + f_2, f_1 - f_2, f_1 f_2, \frac{f_1}{f_2}$ $(f(x) \neq 0, \forall x)$, $\max\{f_1(x), f_2(x)\}$ and $\min\{f_1(x), f_2(x)\}$ are measurable functions.

(iii) Let $f_1, f_2, f_3, \cdots : E \to \mathbb{R}$ be measurable functions, then

$$\sup_{n\geq 1} f_n(x), \quad \inf_{n\geq 1} f_n(x), \quad \overbrace{\overline{\lim_{n\to\infty}} f_n(x), \quad \underline{\lim_{n\to\infty}} f_n(x),}^{\text{provided they exist in } \mathbb{R} \text{ at a.e. } x \in E}$$

$$\lim_{n\to\infty} f_n(x) \text{ (if } f_n \to f \text{ ptwise a.e. on } E)$$

are measurable.

(iv) Let $f,g : A \to \mathbb{R}$ be two functions such that $f = g$ a.e.. If f is measurable, so is g.

(v) Continuous Functions defined on a measurable domain are measurable.

Example 1. Let $A \subseteq \mathbb{R}$, show that

$$\chi_A(x) := \begin{cases} 1 & \text{when } x \in A \\ 0 & \text{when } x \notin A \end{cases} \text{ is measurable } \iff A \text{ is measurable.}$$

Solution. For every $a \in \mathbb{R}$ we have

$$\chi_A^{-1}[a,\infty) = \begin{cases} \emptyset, & a > 1, \\ A, & a \in (0,1], \\ \mathbb{R}, & a \leq 0. \end{cases}$$

Therefore χ_A is measurable if and only if $\emptyset, A$ and $\mathbb{R}$ are all measurable if and only if A is measurable.

Example 2. Let $f_1, f_2, \cdots : E \to \mathbb{R}$ be a sequence of measurable functions which is pointwise bounded, i.e., $\{f_n(x)\}_{n=1}^\infty$ is bounded for every $x \in E$.

Without using M_k/m_k Theorem, show that

$$\varlimsup_{n\to\infty} f_n \quad \text{and} \quad \varliminf_{n\to\infty} f_n$$

are measurable functions on E.

Solution. We check that $\varlimsup f_n$ is measurable by showing that for every $a \in \mathbb{R}$, the set

$$A := \{x \in E : \varlimsup f_n(x) < a\}$$

is measurable. Let $a \in \mathbb{R}$, then for $x \in A$, we try to find another description of x in order to express A in another form. Equivalently, since $x \in A$ iff $\varlimsup f_n(x) < a$, we try to modify the statement that $\varlimsup f_n(x) < a$.

Unsuccessful but Necessary Trial. Specifically, $\varlimsup f_n(x) < a$ implies

$$\exists N \geq 1, \forall n \geq N, f_n(x) < a.$$

Note that the last statement cannot be reversed. Since if we take $\varlimsup$ on both sides, "$<$" becomes "$\leq$". But still we can proceed by modifying the bound a.

Correct Way. We have

$$\begin{aligned}\varlimsup f_n(x) < a &\implies \exists p \in \mathbb{N}, \varlimsup f_n(x) < a - \frac{1}{p} \\ &\implies \exists p \in \mathbb{N}, \exists N \in \mathbb{N}, \forall n \geq N, f_n(x) < a - \frac{1}{p}.\end{aligned}$$

Fortunately the last statement can be reversed to $\varlimsup f_n(x) < a$, so we have

$$\begin{aligned}A &= \{x \in E : \varlimsup f_n(x) < a\} \\ &= \left\{x \in E : \exists p \in \mathbb{N}, \exists N \in \mathbb{N}, \forall n \geq N, f_n(x) < a - \frac{1}{p}\right\} \\ &= \bigcup_{p=1}^\infty \bigcup_{N=1}^\infty \bigcap_{n=N}^\infty \left\{x \in E : f_n(x) < a - \frac{1}{p}\right\} \\ &= \bigcup_{p=1}^\infty \bigcup_{N=1}^\infty \bigcap_{n=N}^\infty f_n^{-1}\left(-\infty, a - \frac{1}{p}\right).\end{aligned}$$

As every union and intersection is countable, by the hypothesis that f_n's are measurable, we are done.

Now $\varliminf f_n = -\varlimsup(-f_n)$ is measurable by the last paragraph.

Exercise 1. Let $f : E \to \mathbb{R}$ be measurable. Show that if A has measure zero, then f is measurable if and only if $f|_{E \setminus A} : E \setminus A \to \mathbb{R}$ is measurable.

Exercise 2 (2012 Final). Let W be a measurable subset of $\mathbb{R}$ with $m(W) > 0$. For $n = 1, 2, 3, \ldots$, let $f_n : W \to \mathbb{R}$ be a measurable function such that

$$\lim_{n\to\infty} f_n(x) \in \mathbb{R} \quad \text{for a.e. } x \in W.$$

Prove that there exists a $c > 0$ and a measurable set $V \subseteq W$ with $m(V) > 0$ such that $x \in V \implies |f_n(x)| \leq c$ for all $n = 1, 2, 3, \ldots$.

Example 3 (2007 Final). Let S be an uncountable set. For every $s \in S$, let $f_s : \mathbb{R} \to [0,1]$ be a continuous function. Define $f(x) = \sup\{f_s(x) : s \in S\}$. Prove that f is measurable.

Solution. Let $a \in \mathbb{R}$, then

$$f(x) < a \iff \sup_{s\in S} f_s(x) < a \implies \forall s \in S, f_s(x) < a.$$

The last statement cannot be reversed to the first since $<$ becomes $\leq$ when taking supremum. Then what to do? Either we shrink the bound a as in Example 2 (left as exercise) or we try $\leq a, > a$ or $\geq a$ instead.

Note that when S is countable the statement is trivial since we have already such a result that

$$f_1, f_2, \ldots \text{ measurable} \implies \sup_{n\geq 1} f_n \text{ measurable.}$$

We expect the proof is a bit different. Thus continuity must be brought into consideration.

Method 1. Let's consider $\leq a$. We have for any $a \in \mathbb{R}$,

$$f(x) \leq a \iff f_s(x) \leq a, \forall s \in S,$$

therefore

$$f^{-1}(-\infty, a] = \bigcap_{s\in S} f_s^{-1} \underbrace{(-\infty, a]}_{\text{closed}},$$

thus $f^{-1}(-\infty, a]$ is an intersection of closed set, it must be closed and hence measurable.

Method 2. Let's consider $> a$, from Supremum Limit Theorem we have

$$f(x) > a \iff \exists s \in S, f_s(x) > a,$$

therefore we have

$$\{x \in \mathbb{R} : f(x) > a\} = \bigcup_{s\in S} \{x \in \mathbb{R} : f_s(x) > a\} = \bigcup_{s\in S} f_s^{-1}(a, \infty).$$

Since $\bigcup_{s\in S} f_s^{-1}(a,\infty)$ is a union of open sets, which is open and hence measurable.

Remark. We wouldn't expect to argue like "$\bigcup_{s\in S}$ (measurable) is measurable" since the union $\bigcup_{s\in S}$ is uncountable.

Exercise 3 (2004 Final). Let W be a nonempty subset of $\mathbb{R}$, define $f : \mathbb{R} \to [0, \infty)$ by letting $f(x)$ be the greatest lower bound of $\{|x - w| : w \in W\}$. Prove that f is measurable.

Example 4 (2005 Final). Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose for every $\epsilon > 0$, there exists a continuous $g : \mathbb{R} \to \mathbb{R}$ such that $S = \{x \in \mathbb{R} : f(x) \neq g(x)\}$ is measurable with $m(S) < \epsilon$. Prove that f is a measurable function.

Remark. The converse of this statement is a famous result in measure theory known as Lusin's Theorem.

Solution. Now for every $n \in \mathbb{N}$ we may set $\epsilon = \frac{1}{n}$, then by hypothesis there is $g_n \in C(\mathbb{R})$ and a measurable A_n such that $f|_{A_n} = g|_{A_n}$ and $m(\mathbb{R} \setminus A_n) < \frac{1}{n}$. We expect $A := \bigcup_{n=1}^\infty A_n$ is so "huge" that $m(\mathbb{R} \setminus A) = 0$. Indeed,

$$\forall n \in \mathbb{N}, \quad m(\mathbb{R} \setminus A) \leq m(\mathbb{R} \setminus A_n) < \frac{1}{n},$$

by taking $n \to \infty$, $m(\mathbb{R} \setminus A) = 0$. Let's show that $f|_A$ is measurable as $\mathbb{R} \setminus A$ is negligible, more precisely, by Exercise 1, f is measurable if and only if $f|_A$ is measurable.

Method 1. We have

$$\{x \in A : f|_A(x) > a\} = \bigcup_{n=1}^\infty \{x \in A_n : g_n|_{A_n}(x) > a\} = \bigcup_{n=1}^\infty \overbrace{\underbrace{\left(g_n^{-1}(a,\infty)\right)}_{\text{open}} \cap A_n}^{\text{measurable}},$$

so $\{x \in A : f|_A(x) > a\}$ is a countable union of measurable sets, which must be measurable.

Method 2. Define $A_1' = A_1$ and $A_n' = A_n \setminus \bigcup_{i=1}^{n-1} A_i$ for $n \geq 2$, then $\{A_n'\}$ is disjoint and $\bigcup_{n=1}^\infty A_n' = \bigcup_{n=1}^\infty A_n = A$, hence

$$f|_A = \sum_{n=1}^\infty f|_{A_n'} \chi_{A_n'} = \sum_{n=1}^\infty g_n \chi_{A_n'},$$

so $f|_A$ is a pointwise limit of measurable functions, and thus measurable.

Example 5 (2009 Final). Let $f : [0,\infty) \to [0,1]$ be measurable. Prove that the set

$$S = \left\{ a \in [0,\infty) : \sum_{i=1}^{\infty} f(a+i) \in \mathbb{R} \right\}$$

is measurable.

Solution. We note that the Cauchy criterion for the convergence of $\sum_{i=1}^{\infty} f(a+i)$ can be written as

$$\forall k \in \mathbb{N}, \exists N \in \mathbb{N}, \forall n \geq N, \forall m \geq n, \sum_{i=n}^{m} f(a+i) < \frac{1}{k}.$$

Therefore

$$\left\{ a \in [0,\infty) : \sum_{i=1}^{\infty} f(a+i) \in \mathbb{R} \right\} = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcap_{m=n}^{\infty} \left\{ a \in [0,\infty) : \sum_{i=n}^{m} f(a+i) < \frac{1}{k} \right\}.$$

Clearly it remains to check the measurability of $f_i(x) := f(x+i)$. For every interval (a,b) we have

$$\begin{aligned} x \in f_i^{-1}(a,b) &\iff f_i(x) = f(x+i) \in (a,b) \\ &\iff x+i \in f^{-1}(a,b) \\ &\iff x \in -i + f^{-1}(a,b), \end{aligned}$$

hence

$$f_i^{-1}(a,b) = -i + f^{-1}(a,b).$$

Since a translation of a measurable set is still measurable, f_i is measurable. Now

$$S = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcap_{m=n}^{\infty} \underbrace{(f_n + f_{n+1} + \cdots + f_m)^{-1} \left[0, \tfrac{1}{k}\right)}_{\text{measurable}}$$

is measurable.

In the following let's slightly generalize the above example. Of course we can copy, word by word, the solution of the above example, let's solve it alternatively:

Exercise 4. Let $f_1, f_2, f_3, \dots : E \to \mathbb{R}$ be a sequence of measurable functions. Show that

$$A := \{x \in E : \{f_n(x)\}_{n=1}^{\infty} \text{ converges}\}$$

is measurable **by taking** $\overline{\lim}_{n\to\infty} f_n$ **and** $\underline{\lim}_{n\to\infty} f_n$ **into account**.

Caution: It is not as simple as it seems to be since $\overline{\lim} f_n(x)$ and $\underline{\lim} f_n(x)$ are possibly unbounded, $\overline{\lim} f_n - \underline{\lim} f_n$ may carry no meaning in this case, moreover, function taking value in $\{-\infty, \infty\}$ is not considered as a measurable function **in this course**.

Example 6. Every real number $x \in (0,1]$ has a unique **nonterminating representation**$^{(\dagger)}$ (we signify it by putting a $_\times$ at the tail)

$$x = 0.a_1 a_2 a_3 \dots_\times .$$

We define a function $f : (0,1] \to \mathbb{R}$ pointwise by

$$f(x) = \sup\{a_k : x = 0.a_1 a_2 a_3 \dots_\times \in (0,1], k \in \mathbb{N}\},$$

show that f is measurable.

Solution. **Method 1.** By Exercise 5, $f(x) = 9$ a.e., and constant function 9 is measurable, so f is measurable.

Method 2. Let

$$A_k = \{x \in (0,1] : x = 0.a_1 a_2 \dots_\times, a_i \geq k, \exists i\}.$$

Then observe that when $f(x) = \ell$, we have

$$x \in A_1, x \in A_2, \dots, x \in A_\ell, x \notin A_{\ell+1}, \dots, x \notin A_9.$$

Then we find that $\sum_{i=1}^{9} \chi_{A_i}(x) = \ell$, therefore we have

$$f = \sum_{i=1}^{9} \chi_{A_i}.$$

It remains to check that each A_i is measurable, we leave it as a practice in Exercise 6.

Exercise 5. Prove the above example by showing that

$$m\left\{ x = 0.a_1 a_2 \dots_\times \in (0,1] : a_i = 9 \text{ for some } i \right\} = 1.$$

Exercise 6. Show that

$$A_k := \left\{ x \in (0,1] : \begin{array}{c} x = 0.a_1 a_2 \dots_\times, \\ \exists i, a_i \geq k \end{array} \right\} = \bigcup_{i=1}^{\infty} \bigcup_{a_i = k}^{9} \bigcup_{j=0}^{10^{i-1}-1} \left(\frac{10j + a_i}{10^i}, \frac{10j + a_i + 1}{10^i} \right].$$

Hint: Imitate the solution of Practice Exercise 80 of lecture notes.