Riemann Integral (Part I): Riemann Integrability

## We need to know

- how to judge whether a function is Riemann integrable or not.

## Key definitions and results

**Definition 1 (Partition, Mesh).** Given a **partition** $P = \{x_0, \ldots, x_n\}$ of $[a,b]$, where $a = x_0 < x_1 < \cdots < x_n = b$, we define $\boldsymbol{\Delta x_i = x_i - x_{i-1}}$ and define the **mesh** of $P$ by $\|P\| = \max_{1\le i\le n} \Delta x_i$.

**Definition 2 (Upper, Lower Sum).** Let $f : [a,b] \to \mathbb{R}$ and $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$, we define the **lower Riemann sum** and **upper Riemann sum** w.r.t. $P$ to be

$$L(f,P) = \sum_{i=1}^{n} \underbrace{\inf\left(f([x_{i-1},x_i])\right)}_{:=m_i} \Delta x_i = \sum_{i=1}^{n} m_i \Delta x_i$$

$$U(f,P) = \sum_{i=1}^{n} \underbrace{\sup\left(f([x_{i-1},x_i])\right)}_{:=M_i} \Delta x_i = \sum_{i=1}^{n} M_i \Delta x_i.$$

Also, given choices $x_i^* \in [x_{i-1}, x_i]$, we define **Riemann sum** of $f$ w.r.t. $P$ to be

$$S(f,P) = \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

**Remark.** The **beginning definition of Riemann integrability** of $f : [a,b] \to \mathbb{R}$ is: there is an $I \in \mathbb{R}$ such that for any $\epsilon > 0$, there is a $\delta > 0$ such that $\|P\| < \delta \implies |S(f,P) - I| < \epsilon$. Immediately from this definition $f(x)$ is bounded (Exercise 1). In the rest of the definitions and results we will require $f(x)$ be bounded, then several useful characterizations of Riemann integrability emerges.

**Definition 3 (Upper, Lower Integral).** Suppose $f : [a,b] \to \mathbb{R}$ is bounded, the **lower integral** of $f(x)$ on $[a,b]$ is the *biggest inner approximation*:

$$\underline{\int_a^b} f(x)\,dx = \sup\left\{L(f,P) : P \text{ is a partition of } [a,b]\right\}$$

and the **upper integral** of $f(x)$ on $[a,b]$ is the *smallest outer approximation*:

$$\overline{\int_a^b} f(x)\,dx = \inf\left\{U(f,P) : P \text{ is a partition of } [a,b]\right\}.$$

**Definition 4 (Integrability).** We say that $f(x)$ is **Riemann integrable** on $[a,b]$ if

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx,$$

and the common value is denoted by $\int_a^b f(x)\,dx$.

**Theorem 5 (Refinement).** For partitions $P, P'$ of $[a,b]$, we say that $P'$ is a **refinement** of $P$ if $P' \supseteq P$, in this case, we have

$$L(f,P) \le L(f,P') \le U(f,P') \le U(f,P).$$

**Theorem 6 (Integral Criterion, Darboux).** Let $f(x)$ be *bounded* on $[a,b]$, then the following are *equivalent*:

(a) $f(x)$ is Riemann integrable on $[a,b]$.

(b) $\forall \epsilon > 0$, $\exists$ a partition $P$ of $[a,b]$, s.t. $U(f,P) - L(f,P) < \epsilon$. **(Integral Criterion Theorem)**

(c) $\exists I$, for every $\epsilon > 0$, there is a $\delta > 0$ s.t. $\|P\| < \delta \implies |S(f,P) - I| < \epsilon$. **(Riemann's Original Approach)**

**Theorem 7.** Let $f(x)$ be continuous on $[a,b]$, then $f(x)$ is integrable on $[a,b]$.

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**Remark.** We will use Integral Criterion very frequently, for the sake of notational simplicity let's define

$$(U-L)(f,P) = U(f,P) - L(f,P),$$

then the **integrability of $f$** is the same as

$$\forall \epsilon > 0, \exists \text{ a partiaion } P \text{ of } [a,b] \quad \text{s.t.} \quad (U-L)(f,P) < \epsilon. \qquad (*)$$

Or **equivalently**,

$$\forall \epsilon > 0, \exists \delta > 0, \|P\| < \delta \implies (U-L)(f,P) < \epsilon. \qquad (**)$$

$(*)$ and $(**)$ are equivalent, and using which one of them depends on whether the **mesh** $\|P\|$ needs to be considered.

**Example 1 (Increasing Functions & Dirichlet Function).**

(a) Show that if $f : [a,b] \to \mathbb{R}$ is increasing, then $f(x)$ is Riemann integrable.

(b) Show that the Dirichlet's function $D : \mathbb{R} \to \mathbb{R}$ given by

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is **not** Riemann integrable on any closed interval.

Sol **(a)** For any $P \sqsubset [a,b]$, we have

$$\begin{aligned} (U-L)(f,P) &:= U(f,P) - L(f,P) \\ &= \sum \Big( \sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \Big) \Delta x_i \\ &= \sum (f(x_i) - f(x_{i-1})) \Delta x_i \\ &\le \sum (f(x_i) - f(x_{i-1})) \|P\| \\ &= (f(b) - f(a)) \|P\|. \end{aligned}$$

Therefore for every $\epsilon > 0$, we can choose $P \sqsubset [a,b]$ such that

$$\|P\| < \epsilon / (f(b) - f(a) + 1),$$

with this kind of partition,

$$(U-L)(f,P) < \epsilon,$$

therefore $f(x)$ is Riemann integrable on $[a,b]$ by Integral Criterion.

**(b)** For any $P \sqsubset [a,b]$, we have

$$U(D,P) = \sum \Big( \sup_{[x_{i-1},x_i]} D \Big) \Delta x_i = \sum 1 \cdot \Delta x_i = b - a,$$

and we have

$$L(D,P) = \sum \Big( \inf_{[x_{i-1},x_i]} D \Big) \Delta x_i = \sum 0 \cdot \Delta x_i = 0,$$

therefore

$$(U-L)(D,P) = U(D,P) - L(D,P) = b - a =: \epsilon_0,$$

we conclude there is $\epsilon_0$, for any $P \sqsubset [a,b]$, $(U-L)(D,P) \ge \epsilon_0$ (the negation of Integral Criterion), so $D(x)$ cannot be integrable over any closed subinterval. ∎

**Example 2.** Let $f(x), h(x)$ be Riemann integrable on $[a,b]$ and let $g : [a,b] \to \mathbb{R}$ be such that

$$f(x) \le g(x) \le h(x) \quad \text{for all } x \in [a,b].$$

Show that if $\displaystyle\int_a^b f(x)\,dx = A = \int_a^b h(x)\,dx$, then $g(x)$ is also Riemann integrable.

Sol For every $P \sqsubset [a,b]$, we have

$$L(f,P) \le L(g,P) \le U(g,P) \le U(h,P). \qquad (!)$$

The Riemann integrability of $f(x)$ and $h(x)$ says that $L(f,P)$ and $U(h,P)$ is close to $A$ when $P$ is refined enough, and then $(U-L)(g,P)$ will be forced to be very close to zero, let's make this precise now.

**Method 1.** Note that the Riemann integrability of $f : [a,b] \to \mathbb{R}$ says that for every $\epsilon > 0$, there is $\delta > 0$ such that

$$\|P\| < \delta \implies \left| S(f,P) - \int_a^b f(x)\,dx \right| < \epsilon$$

for every choices $x_i^* \in [x_{i-1}, x_i]$ $(i = 1, 2, \dots)$. By taking infimum and supremum over all $x_i^* \in [x_{i-1}, x_i]$ for each $i$, we have

$$\|P\| < \delta \implies \left| L(f,P) - \int_a^b f(x)\,dx \right| \le \epsilon \quad \text{and} \quad \left| U(f,P) - \int_a^b f(x)\,dx \right| \le \epsilon.$$

The same is true when $f$ is replaced by $h$ and with $\delta$ replaced by $\delta'$. Take $\rho = \min\{\delta, \delta'\}$, then by (!),

$$\|P\| < \rho \implies A - \epsilon \le L(g,P) \le U(g,P) \le A + \epsilon,$$

and we are done. ∎

**Method 2.** Let $\epsilon > 0$ be given. As $\int_a^b f(x)\,dx = A = \underline{\int_a^b} f(x)\,dx = \sup\{L(f,P) : P \sqsubset [a,b]\}$, there is a partition $P_1 \sqsubset [a,b]$ such that

$$A - L(f, P_1) < \epsilon.$$

Since $\int_a^b h(x)\,dx = A = \overline{\int_a^b} h(x)\,dx = \inf\{U(h,P) : P \sqsubset [a,b]\}$, there is a partition $P_2 \sqsubset [a,b]$ such that

$$U(h, P_2) - A < \epsilon.$$

Now we get a partition $P = P_1 \cup P_2$ refining both $P_1$ and $P_2$, then by Refinement Theorem,

$$A - \epsilon < L(f, P_1) \le L(f,P) \quad \text{and} \quad U(h,P) \le U(h,P_2) < A + \epsilon,$$

therefore from (!),

$$A - \epsilon < L(g,P) \le U(g,P) < A + \epsilon,$$

and thus

$$(U-L)(g,P) < (A+\epsilon) - (A-\epsilon) = 2\epsilon. \qquad ∎$$

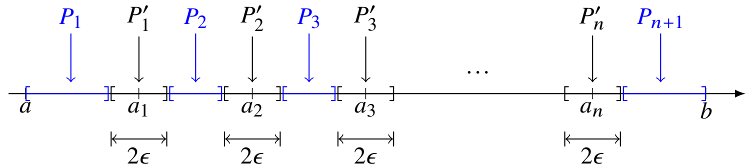
**Example 3.** Let $f(x)$ be Riemann integrable on $[a,b]$. Show that $g(x)$ on $[a,b]$ defined by

$$g(x)=\begin{cases}1 & \text{if } x=a_1,a_2,\dots,a_n\\ f(x) & \text{if } x\in[a,b]\setminus\{a_1,a_2,\dots,a_n\}\end{cases}$$

is Riemann integrable, where $a_1,a_2,\dots,a_n\in(a,b)$.

**Remark.** We list a similar past exam problem in Exercise 3.

Sol For every $\epsilon>0$, we take the following partition and denote $P_i,P_i'$ the to be chosen partition of each subinterval.




That is, we first draw $n$ small intervals to cover those $a_1,\dots,a_n$. Next, in the remaining $n+1$ intervals we choose $P_i$ such that for each $i$,

$$(U-L)(f,P_i)<\epsilon$$

Let $P_i'=\{a_1-\epsilon,a_1+\epsilon\}$, i.e., the end points of the interval we use to cover $a_i$'s, then for every $i$,

$$(U-L)(f,P_i')=\left(\sup_{[a_i-\epsilon,a_i+\epsilon]}f-\inf_{[a_i-\epsilon,a_i+\epsilon]}f\right)\cdot(2\epsilon)\le 2\sup_{[a,b]}|f|\cdot 2\epsilon=4\sup_{[a,b]}|f|\cdot\epsilon.$$

Since $f(x)$ is Riemann integrable on $[a,b]$, it is automatically bounded on $[a,b]$, so $\sup_{[a,b]}|f|<\infty$. Therefore if we take

$$P=P_1\cup P_1'\cup P_2\cup P_2'\cup\cdots\cup P_{n+1},$$

then $P\sqsubset[a,b]$ and

$$\begin{aligned}(U-L)(f,P)&=\sum_{i=1}^{n+1}(U-L)(f,P_i)+\sum_{i=1}^{n}(U-L)(f,P_i')\\&<(n+1)\epsilon+4n\epsilon\sup_{[a,b]}|f|=\left(n+1+4n\sup_{[a,b]}|f|\right)\epsilon.\end{aligned}$$

Since all the numbers—$n$ and $\sup_{[a,b]}|f|$—are fixed (absolutely a constant), by Integral Criterion we are done. ∎

**Example 4 (2008 Spring).** Let $f_1,f_2,\dots:[0,1]\to[0,1]$ be Riemann integrable, prove that $g(x)$ on $[0,1]$ given by $g(0)=0$ and

$$g(x)=f_n(x)\quad\text{for } x\in\left(\frac{1}{n+1},\frac{1}{n}\right]$$

is also Riemann integrable *by using Integral Criterion*.

Sol Let $P=\{0,\frac{1}{k+1}\}\cup P'\sqsubset[0,1]$, where $P'\sqsubset[\frac{1}{k+1},1]$, then we have

$$(U-L)(g,P)=\underbrace{(U-L)\left(g,\left\{0,\frac{1}{k+1}\right\}\right)}_{=\left(\sup_{[0,\frac{1}{k+1}]}g-\inf_{[0,\frac{1}{k+1}]}g\right)\cdot\frac{1}{k+1}}+(U-L)(g,P')\le\frac{1}{k+1}+(U-L)(g,P').$$

**Let's fix an $\epsilon>0$.**

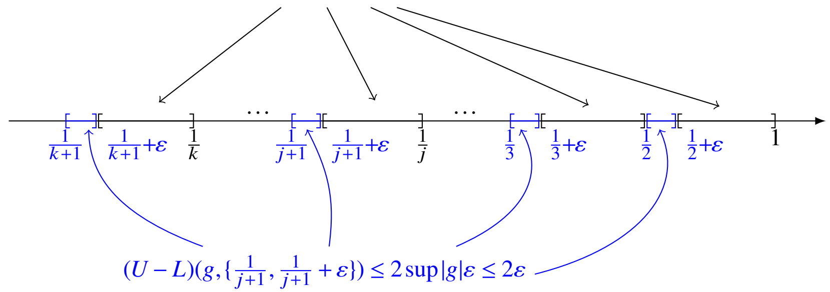
We can fix a $k$ such that $1/(k+1)<\epsilon$, and then

$$(U-L)(g,P)<\epsilon+(U-L)(g,P').$$

It is enough to choose nice enough partition $P'\sqsubset[\frac{1}{k+1},1]$ to make $(U-L)(g,P')$ small (say, $<C\epsilon$ for some absolute constant $C$).

Since $g$ is defined "intervalwise" on $(\frac{1}{j+1},\frac{1}{j}]$'s, let's take $\varepsilon>0$ very very small such that $[\frac{1}{j+1}+\varepsilon,\frac{1}{j}]\subseteq(\frac{1}{j+1},\frac{1}{j}]$ for every $j=1,2,\dots,k$. We try to find partitions $P_1,P_2,\dots,P_k$ such that

$P_j\sqsubset[\frac{1}{j+1}+\varepsilon,\frac{1}{j}]$ with $(U-L)(g,P_j)=(U-L)(f_j,P_j)$ small, and




(recall that $\{\frac{1}{j+1},\frac{1}{j+1}+\varepsilon\}$ is also a partition of $[\frac{1}{j+1},\frac{1}{j+1}+\varepsilon]$) Taking union over all these partitions, we get a partition of $[\frac{1}{k+1},1]$ that makes all $(U-L)(g,\cdot)$'s small.

Let's find such $P_j$'s. By the Riemann integrability of $f_j$ restricted to a subinterval, for the $\epsilon>0$ above we can find $P_j\sqsubset[\frac{1}{j+1}+\varepsilon,\frac{1}{j}]$ such that

$$(U-L)(g,P_j)\overset{(!)}{=\!=}(U-L)(f_j,P_j)<\frac{\epsilon}{2^j}.$$

The equality (!) holds since $g = f_j$ completely on $[\frac{1}{j+1}+\varepsilon, \frac{1}{j}]$ (but not on $[\frac{1}{j+1}, \frac{1}{j}]$, that's why we shrink the interval before we apply Riemann integrability).

Now

$$P' := \bigcup_{j=1}^{k} \Big( \underbrace{\Big\{ \frac{1}{j+1}, \frac{1}{j+1}+\varepsilon \Big\} \cup P_j}_{\subset [\frac{1}{j+1}, \frac{1}{j}]} \Big) \subset \Big[\frac{1}{k+1}, 1\Big],$$

and therefore we have

$$\begin{aligned}(U-L)(g,P') &= \sum_{j=1}^{k} \Big( (U-L)(g, \{\tfrac{1}{j+1}, \tfrac{1}{j+1}+\varepsilon\}) + (U-L)(g, P_j) \Big) \\ &\le \sum_{j=1}^{k} \Big( 2\varepsilon + \frac{\epsilon}{2^j} \Big) < 2k\varepsilon + \epsilon.\end{aligned}$$

We may take $\varepsilon = \frac{\epsilon}{2k}$ at the beginning to conclude $(U-L)(g,P') < 2\epsilon$. ∎

> **Remark.** The solutions of Example 3 and Example 4 can be much much simpler with the help of Lebesgue Criterion in the next tutorial note. The difficult work here serve as examples for us to appreciate Lebesgue Criterion.

# Exercise

**1.** The beginning definition of **Riemann integrability** of $f : [a,b] \to \mathbb{R}$ is: there is an $I \in \mathbb{R}$, for any $\epsilon > 0$, there is a $\delta > 0$ s.t. $\|P\| < \delta \implies |S(f,P) - I| < \epsilon$.

Suppose that $f(x)$ is Riemann integrable on $[a,b]$. We show that $f(x)$ is bounded as follows:

(a) Let $\epsilon = 1$ and $\delta > 0$ be the corresponding quantity in the definition above. Fix a partition $P = \{x_0, \dots, x_n\}$ of $[a,b]$ s.t. $\|P\| < \delta$. Show that for every $x_1^* \in [x_0, x_1]$, and every **fixed** $x_i^* \in [x_{i-1}, x_i]$, $2 \le i \le n$, we have

$$|f(x_1^*)|\Delta x_1 \le 1 + |I| + \sum_{i=2}^{n} |f(x_i^*)|\Delta x_i.$$

Therefore $f(x)$ is bounded on $[x_0, x_1]$.

(b) Similarly conclude that $f(x)$ is bounded on $[x_1,x_2], [x_2,x_3], \dots, [x_{n-1},x_n]$ respectively and conclude that $f(x)$ is bounded on $[a,b]$.

**Remark.** Therefore when we say that a function is Riemann integrable on $[a,b]$, it is **automatically bounded** on $[a,b]$.

**2.** We know that if $f(x)$ is Riemann integrable on $[0,1]$, then so is $|f(x)|$. The converse is not true in general:

(a) Give an example $f(x)$ on $[a,b]$ that is discontinuous everywhere but $|f|$ is Riemann integrable.

(b) Give an injective function on $[0,1]$ which is not Riemann integrable.

**3.** **(2010 Spring)** Let $f : [0,1] \to [0,1]$ be Riemann integrable. Let $\{r_n\}$ be a strictly increasing sequence in $(0,1)$. Prove that $g : [0,1] \to [0,1]$ defined by

$$g(x) = \begin{cases} 1 & \text{if } x \in \{r_n : n \in \mathbb{N}\} \\ f(x) & \text{if } x \in [0,1] \setminus \{r_n : n \in \mathbb{N}\} \end{cases}$$

is Riemann integrable.

**4.** Let $f(x)$ be Riemann integrable on $[a,b]$, show that if $\displaystyle\int_a^b f(x)\,dx > 0$, then there is an $\eta > 0$ and a closed subinterval $I$ such that $f(x) > \eta$ on $I$.

**5.** Let $f$ be strictly increasing on $[a,b]$. Let $P$ be a partition on $[a,b]$, then $Q := f(P)$ will be a partition of $[f(a), f(b)]$.

(a) Explain why $f^{-1}$ must be Riemann integrable.

(b) Show that $bf(b) - af(a) = U(f,P) + L(f^{-1}, Q)$.

(c) Deduce that

$$bf(b)-af(a) \le \int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(x)\,dx.$$

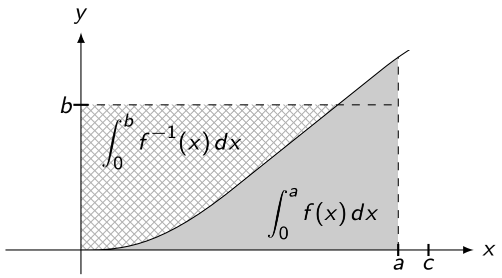
Explain why we have equality when $f$ is continuous.

**6. (Young's Inequality)** Fix a $c>0$, suppose that $f:[0,c]\to\mathbb{R}$ is strictly increasing with $f(0)=0$.

(a) By using Exercise 5, show that whenever $a\in[0,c]$ and $b\in[0,f(c)]$,

$$ab \le \int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx. \qquad (*)$$

**Remark.** This inequality has a very strong geometrical intuition:




We have to avoid any graphical reasoning in our rigorous proof. Also by the same reasoning we allow $ab\ge 0$ and $ab<0$, try to convince yourself this is true by extending the graph below the $y$-axis ☺.

(b) Suppose further that $f(x)$ is continuous on $[0,c]$, show that equality in $(*)$ holds if and only if $b=f(a)$.

**Hint:** Recall Continuous Inverse Theorem.

(c) From (a), deduce that for any $a,b\ge 0$ and $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$,

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q.$$

**7.** Let $k\in\mathbb{N}$ and let $f:[0,\infty)\to[0,\infty)$ be a continuous function such that $f(f(x))=x^k$ for every $x\in[0,\infty)$. Show that

$$\int_0^1 f(x)^2\,dx \ge \frac{2k-1}{k^2+6k-3}.$$

**Hint:** Use Exercise 6.