Lebesgue Inner, Outer Measures and Lebesgue Measure

Key Definitions and Results

Definition 1.

(i) The **Lebesgue outer measure** of a subset $A \subseteq \mathbb{R}$ is

$$m^*(A) = \inf\{\lambda(U) : U \supseteq A,\ U \text{ open}\}.$$

(ii) The **Lebesgue inner measure** of a subset $A \subseteq \mathbb{R}$ is

$$m_*(A) = \sup\{\lambda(K) : K \subseteq A,\ K \text{ compact}\}.$$

(iii) A bounded set A is said to be **Lebesgue measurable** if $m_*(A) = m^*(A)$ and the common value is the **Lebesgue measure**, denoted by $m(A)$.

(iv) An unbounded set A is said to be **Lebesgue measurable** if $A \cap [a,b]$ is measurable for every $a \le b$. In this case the **Lebesgue measure** of A is

$$m(A) = \lim_{x\to+\infty} m(A \cap [-x,x]).$$

Definition 2. Let $\{E_k\}_{k=1}^\infty$ be a countable collection of subsets of $\mathbb{R}$.

(i) $\{E_k\}$ is **ascending** if $E_k \subseteq E_{k+1}$ for each k, in this case we define

$$\lim_{k\to\infty} E_k = \bigcup_{k=1}^\infty E_k.$$

(ii) $\{E_k\}$ is **descending** if $E_k \supseteq E_{k+1}$ for each k, in this case we define

$$\lim_{k\to\infty} E_k = \bigcap_{k=1}^\infty E_k.$$

Theorem 3 (Existence of Nonmeasurable Sets). Any subset $E \subseteq \mathbb{R}$ with positive Lebesgue outer measure contains a nonmeasurable subset.

Theorem 4. Intervals are measurable whose measure are their length.

Theorem 5. Lebesgue measurable sets have the following properties.

(i) If E is measurable, so is $\mathbb{R} \setminus E$.

(ii) If $A_1, A_2, \dots$ are measurable, so is $\bigcup_{i=1}^\infty A_i$.

(iii) If $A_1, A_2, \dots$ are measurable, so is $\bigcap_{i=1}^\infty A_i$.

Theorem 6 (Properties of Outer Measure). The outer and inner measures have the following properties:

(i) $m^*(\emptyset) = 0$.

(ii) $A \subseteq B \implies m_*(A) \le m_*(B)$ and $m^*(A) \le m^*(B)$. **(Monotone)**

(iii) $m^*\left(\bigcup_{i=1}^\infty A_i\right) \le \sum_{i=1}^\infty m^*(A_i)$. **(Subadditive)**

Theorem 7 (Completeness).

(i) If $m^*(A) = 0$, then A is measurable with $m(A) = 0$.

(ii) Any subset of a set of measure zero is measurable with measure zero.

Theorem 8 (Properties of Lebesgue Measure). Let $A, B, E_1, E_2, \dots$ be measurable subsets of $\mathbb{R}$.

(i) $A \subseteq B \implies m(A) \le m(B)$. **(Monotone)**

(ii) $x \in \mathbb{R} \implies A + x$ is measurable and $m(A+x) = m(A)$. **(Translation Invariant)**

(iii) $m\left(\bigcup_{n=1}^\infty E_n\right) \le \sum_{n=1}^\infty m(E_n)$. **(Subadditive)**

(iv) E_n's pairwise disjoint $\implies m\left(\bigcup_{n=1}^\infty E_n\right) = \sum_{n=1}^\infty m(E_n)$. **(Countably Additive)**

Theorem 9 (Monotone Set).

(i) If $\{A_k\}_{k=1}^\infty$ is an ascending collection of measurable sets, then

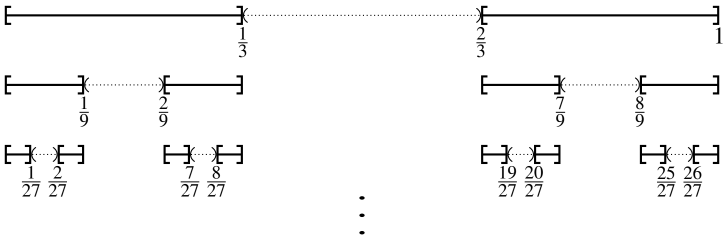
$$m\left(\bigcup_{k=1}^\infty A_k\right) := m\left(\lim_{k\to\infty} A_k\right) = \lim_{k\to\infty} m(A_k).$$

(ii) If $\{B_k\}_{k=1}^\infty$ is a descending collection of measurable sets and $\boldsymbol{m(B_N) < \infty}$, **for some $N \in \mathbb{N}$**, then

$$m\left(\bigcap_{k=1}^\infty B_k\right) := m\left(\lim_{k\to\infty} B_k\right) = \lim_{k\to\infty} m(B_k).$$

Example 1. Show that Cantor set C has measure zero.

Solution.



Denote the Cantor set in the nth stage by C_n. We note that

$$m(C_n) = 2^n \times \frac{1}{3^n}.$$

As $\{C_n\}$ is descending, we have

$$C = \bigcap_{n=1}^{\infty} C_n =: \lim_{n\to\infty} C_n.$$

As $\boldsymbol{m(C_1) \leq 1 < \infty}$, by Monotone Set Theorem we have

$$m(C) = m\left(\lim_{n\to\infty} C_n\right) = \lim_{n\to\infty} m(C_n) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0.$$

Example 2. Let E_1 and E_2 be two measurable subsets of $\mathbb{R}$ that have finite measure, show that

$$m(E_1 \cup E_2) = m(E_1) + m(E_2) - m(E_1 \cap E_2).$$

Solution. Recall that for any set $A, B \subseteq \mathbb{R}$ we have

$$A = (A \cap B) \sqcup (A \setminus B).$$

That is, the set B and its complement B^c can be used to split A, and vice versa. Therefore when A, B are measurable, Countable Additivity of Lebesgue measure tells us

$$m(A) = m(A \cap B) + m(A \setminus B).$$

Replacing A by $E_1 \cup E_2$ and B by E_1, we have

$$\begin{aligned} m(E_1 \cup E_2) &= m((E_1 \cup E_2) \cap E_1) + m((E_1 \cup E_2) \setminus E_1) \\ &= m(E_1) + m(E_2 \setminus E_1). \end{aligned}$$

We repeat the process to get

$$m(E_2) = m(E_2 \cap E_1) + m(E_2 \setminus E_1),$$

combining them to eliminate $m(E_2 \setminus E_1)$, we are done.

Exercise 1. Define $A \Delta B = (A \setminus B) \cup (B \setminus A)$, show that if $A, B \subseteq \mathbb{R}$ are measurable,

$$m(A \Delta B) = 0 \implies m(A) = m(B).$$

Example 3. Let $E \subseteq \mathbb{R}$ be uncountable and $C \subseteq [0,1]$ the Cantor set. Suppose that for every $e \in E$, there is a $q \in \mathbb{Q}$ such that $e+q \in C$, show that E is measurable.

Solution. We use the condition on E to obtain set containment. Let $e \in E$, then $\exists q \in \mathbb{Q}$, $e+q \in C$, i.e., $\exists q \in \mathbb{Q}$, $e \in C-q$, so $e \in \bigcup_{q\in\mathbb{Q}}(C-q)$. This is true for each $e \in E$, thus

$$E \subseteq \bigcup_{q\in\mathbb{Q}} (C-q).$$

By Subadditivity of outer measure we have

$$m^*(E) \le \sum_{\underbrace{q\,\in\,\mathbb{Q}}_{\textbf{countable}}} m^*(C-q) = \sum_{q\in\mathbb{Q}} m(C-q) = \sum_{q\in\mathbb{Q}} m(C) = 0. \tag{1}$$

Therefore $m^*(E) = 0$ and hence E is measurable with $m(E) = 0$.

Remark. In (1) we cannot drop the $*$ in $m^*(E)$ as it is not known that whether the set E is measurable, in fact we don't have an explicit formula for E. More precisely, any subset of $\bigcup_{q\in\mathbb{Q}}(C-q)$ can be chosen to be the "E" in this example.

Remark. In (1) we have used that $m^*(C-q) = m(C-q) = m(C)$, in fact we can also say that

$$m^*(C-q) = m^*(C) = m(C)$$

since m^* is also translation invariant, no matter the set C itself is measurable or not. In fact, m is translation invariant due to the fact that m^* does (can you prove this? consider outer approximation of any sets by definition of outer measure).

Exercise 2. Let $E \subseteq \mathbb{R}$ be such that $m^*(E) > 0$. Show that E contains a bounded subset with positive outer measure.

Exercise 3. Let $E \subseteq \mathbb{R}$. Suppose for each $x \in E$ there is an open interval $(x-\delta_x, x+\delta_x)$ such that

$$m^*\big(E \cap (x-\delta_x, x+\delta_x)\big) = 0,$$

show that $m^*(E) = 0$.

If we further assume E is measurable, show that $m(E) = 0$ alternatively by using inner approximation by compact sets and finite covering arguments.

Example 4. Let $E_1, E_2, \ldots$ be a sequence of measurable subsets of $\mathbb{R}$ with $m(E_n) = 0$ for each $n \in \mathbb{N}$. Let

$$H_1 = \{x \in \mathbb{R} : x \text{ lies in at least 1 of } E_n\text{'s}\}$$
$$H_2 = \{x \in \mathbb{R} : x \text{ lies in EXACTLY 1 of } E_n\text{'s}\},$$

show that H_1, H_2 are all measurable and $m(H_1) = m(H_2) = 0$.

Solution. Easy to see that $H_1 = \bigcup_{n=1}^\infty E_n$. Since H_1 is a countable union of measurable sets, H_1 is measurable. Subadditivity of Lebesgue measure yields

$$m(H_1) \le \sum_{n=1}^\infty m(E_n) = \sum_{n=1}^\infty 0 = 0,$$

so $m(H_1) = 0$.

Next consider H_2, we note that

$$\begin{aligned} x \in H_2 &\iff x \in H_i, \exists! i \\ &\iff \exists i, \forall j \ne i, x \in H_i, x \notin H_j, \end{aligned}$$

so we have

$$H_2 = \bigcup_{i=1}^\infty \bigcap_{\substack{j=1\\ j\ne i}}^\infty (H_i \setminus H_j) = \underbrace{\bigcup_{i=1}^\infty \Bigg(\overbrace{H_i \setminus \overbrace{\bigcup_{\substack{j=1\\ j\ne i}}^\infty H_j}^{\text{measurable}}}^{\text{measurable}} \Bigg)}_{\text{measurable}}.$$

Finally, since $H_2 \subseteq H_1$, we have $m(H_2) = 0$ by either Subadditivity or Monotonicity of Lebesgue measure.

Exercise 4 (2005 Final). Prove that the intersection of measurable subsets in $\mathbb{R}$ can be a nonmeasurable set in $\mathbb{R}$.

Exercise 5. Show that if K is compact and L is closed, then

$$K + L := \{k + l : k \in K, l \in L\}$$

is closed by using the sequential criterion in Example 3 of tutorial note 7, therefore $K+L$ is measurable in this case. Is $K+L$ still measurable if compactness of K is replaced by closedness?

Example 5. [2005 Final] Let $E_1, E_2, \ldots, E_k \subseteq [0,1]$ be measurable such that $\sum_{i=1}^{k} m(E_i) > k-1$, prove that $m\left(\bigcap_{i=1}^{k} E_i\right) > 0$.

Solution. We understand union (due to the Subadditivity) more than intersection, so let's translate the quantity in the following way:

$$1 - m\left(\bigcap_{i=1}^{k} E_i\right) = m\left([0,1] \setminus \bigcap_{i=1}^{k} E_i\right) = m\left(\bigcup_{i=1}^{k}([0,1] \setminus E_i)\right).$$

By Subadditivity we have

$$1 - m\left(\bigcap_{i=1}^{k} E_i\right) \le \sum_{i=1}^{k} m([0,1] \setminus E_i) = \sum_{i=1}^{k}(1 - m(E_i)) = k - \sum_{i=1}^{k} m(E_i).$$

Since $\sum_{i=1}^{k} m(E_i) > k-1$, we obtain

$$1 - m\left(\bigcap_{i=1}^{k} E_i\right) < k - (k-1) = 1,$$

therefore $m(\bigcap_{i=1}^{k} E_i) > 0$.

Exercise 6 (2005 Final (Version 2)). Let $E_1, E_2, E_3, \cdots \subseteq [0,1]$ be measurable such that $\lim_{k\to\infty} m(E_k) = 1$. Prove that there is a subsequence $E_{k_1}, E_{k_2}, E_{k_3}, \ldots$ of E_k's such that $m\left(\bigcap_{n=1}^{\infty} E_{k_n}\right) > \frac{1}{2}$.

The next two exercises focus on the <u>outer regularity</u> of Lebesgue measure. Try to approximate the length of measurable sets from outside by open sets.

Exercise 7 (2003 Final). Let E be a bounded measurable set in $\mathbb{R}$ such that $m(E \cap I) \le \frac{1}{2} m(I)$ for every interval I. Prove that $m(E) = 0$.

Exercise 8. Let E be measurable and define $cE := \{xe : e \in E\}$, show that

$$m(cE) = |c| m(E).$$

You are given that when E is measurable, so is cE.

Example 6 (2010 Final). Let

$$P = \{x \in [0,1] : \text{in } x = 0.a_1 a_2 a_3 \ldots_{[10]},\ a_i\text{'s are prime}\}^{(*)}.$$

Show that P is measurable and compute $m(P)$.

Solution. Let

$$P_n = \{0.a_1 \ldots a_n \ldots \in [0,1] : a_1, \ldots, a_n \text{ are prime}\}.$$

Then we have

$$P = \bigcap_{n=1}^{\infty} P_n,$$

to show P is measurable, it is enough to show each P_n is measurable.

Let's denote $\mathcal{P} = \{2,3,5,7\}$ the set of primes, for simplicity. Note that

$$\begin{aligned} P_n &= \bigcup_{(a_1,\ldots,a_n)\in\mathcal{P}^n} \{0.a_1 \ldots a_n \ldots : a_{n+1}, a_{n+2}, \cdots \in \{0,\ldots,9\}\} \\ &= \bigcup_{(a_1,\ldots,a_n)\in\mathcal{P}^n} [0.a_1 \ldots a_n, 0.a_1 \ldots a_{n-1}(a_n+1)]. \end{aligned} \quad (2)$$

So each P_n is a union of finitely many intervals, P_n is measurable, so is P.

Let's compute $m(P)$. Since P_n is descending, we have

$$P = \bigcap_{n=1}^{\infty} P_n =: \lim_{n\to\infty} P_n.$$

Since $\boldsymbol{m(P_1) < \infty}$, by Montone Set Theorem we obtain

$$m(P) = m\left(\lim_{n\to\infty} P_n\right) = \lim_{n\to\infty} m(P_n).$$

On the other hand, by Subadditivity we have

$$\begin{aligned} m(P_n) &\le \sum_{(a_1,\ldots,a_n)\in\mathcal{P}^n} m([0.a_1 \ldots a_n, 0.a_1 \ldots a_{n-1}(a_n+1)]) \\ &= \sum_{(a_1,\ldots,a_n)\in\mathcal{P}^n} \frac{1}{10^n} = \frac{2^n}{5^n}. \end{aligned}$$

So $\lim_{n\to\infty} m(P_n) = 0$, i.e., $m(P) = 0$.