Math3033 (Fall 2013-2014) **Tutorial Note 7**

Set Operations with Functions;
Topology on $\mathbb{R}$: Openness, Closedness and Compactness

Key Definitions and Results

Definition 1. Let $S \subseteq \mathbb{R}$, an element $s \in S$ is said to be an **interior point of S** if there is $r > 0$ such that $B(s,r) := (s-r, s+r) \subseteq S$.

Definition 2. In $\mathbb{R}$:

(i) A set U is **open** if every $u \in U$ is an interior point.

(ii) A set L is **closed** if $\mathbb{R} \setminus L$ is an open set.

(iii) A set K is **compact** if K is both closed and bounded$^{(*)}$.

Remark. By (b) of Example 1: a set U is open if and only if $\mathbb{R} \setminus U$ is closed.

Theorem 3. Let $f : A \to B$ be a function:

- $S_\alpha \subseteq T_\alpha \implies \bigcup_\alpha S_\alpha \subseteq \bigcup_\alpha T_\alpha$
- $A \subseteq B \implies A = B \setminus (B \setminus A)$
- $S_\alpha \subseteq T_\alpha \implies \bigcap_\alpha S_\alpha \subseteq \bigcap_\alpha T_\alpha$
- $A \setminus \bigcup_\alpha S_\alpha = \bigcap_\alpha (A \setminus S_\alpha)$
- $A \setminus \bigcap_\alpha S_\alpha = \bigcup_\alpha (A \setminus S_\alpha)$
- $A \cap \left(\bigcup_\alpha S_\alpha\right) = \bigcup_\alpha (A \cap S_\alpha)$
- $A \cup \left(\bigcap_\alpha S_\alpha\right) = \bigcap_\alpha (A \cup S_\alpha)$
- $X \subseteq A \implies f(A \setminus X) \supseteq f(A) \setminus f(X)$
- $Y \subseteq B \implies f^{-1}(B \setminus Y) = f^{-1}(B) \setminus f^{-1}(Y)$
- $U \subseteq V \subseteq A \implies f(U) \subseteq f(V)$
- $X \subseteq Y \implies f^{-1}(X) \subseteq f^{-1}(Y)$
- $E \subseteq f^{-1}(f(E))$
- $f(f^{-1}(F)) \subseteq F$
- $f\left(\bigcup_\alpha S_\alpha\right) = \bigcup_\alpha f(S_\alpha)$
- $\boldsymbol{f\left(\bigcap_\alpha S_\alpha\right) \subseteq \bigcap_\alpha f(S_\alpha)}$
- $f^{-1}\left(\bigcup_\alpha S_\alpha\right) = \bigcup_\alpha f^{-1}(S_\alpha)$
- $f^{-1}\left(\bigcap_\alpha S_\alpha\right) = \bigcap_\alpha f^{-1}(S_\alpha)$

$^{(*)}$ A set S is said to be bounded if there is a constant M such that $|s| \le M$ for every $s \in S$.

Theorem 4 (Structure Theorem of Open Sets). The following are equivalent:

(i) A set $U \subseteq \mathbb{R}$ **is open**.

(ii) U **is a countable union of pairwise disjoint open intervals**. Moreover, the decomposition is unique.

Theorem 5 (Topological Properties of Open Sets).

(i) $\emptyset$ and $\mathbb{R}$ are open sets.

(ii) The union of any collection of open sets is an open set.

(iii) The intersection of finitely many open sets is an open set.

Theorem 6 (Topological Properties of Closed Sets).

(i) $\emptyset$ and $\mathbb{R}$ are closed.

(ii) The intersection of any collection of closed sets is closed.

(iii) The union of finitely many closed sets is closed.

Theorem 7 (Topological Continuity). Let $A, B \subseteq \mathbb{R}$, the following are equivalent:

(i) $f : A \to B$ is continuous.

(ii) For every open set U, $f^{-1}(U) = V \cap A$, for some V open.

(iii) For every closed set L, $f^{-1}(L) = K \cap A$, for some K closed.

Extra Definitions and Results

The following have nothing to do with final examination.

Definition 8. A family of subsets of $\mathbb{R}$, $\mathcal{U} = \{U_\alpha : \alpha \in A\}$, is said to be an **open cover** of $E \subseteq \mathbb{R}$ if $\boxed{\text{each } U_\alpha \text{ is open in } \mathbb{R};}$ and $\boxed{\bigcup_{\alpha \in A} U_\alpha \supseteq E.}$ A **finite subcover** of an open cover $\mathcal{U}$ of $A \subseteq \mathbb{R}$ is a finite subset $\{U_1, \dots, U_n\}$ of $\mathcal{U}$ such that $\bigcup_{i=1}^n U_i \supseteq A$.

Theorem 9. $K \subset \mathbb{R}$ is compact $\iff$ any open cover of K has a finite subcover$^{(\dagger)}$.

$^{(\dagger)}$ We also say that any open cover of K can be **thinned into** a finite subcover

Example 1.

(a) Consider the closed interval $[0,1]$.

(i) Show that every point in $(0,1)$ is an interior point of $[0,1]$ and;

(ii) Both 0 and 1 are not interior point of $[0,1]$.

(b) Show that a set $U \subseteq \mathbb{R}$ is open if and only if $\mathbb{R} \setminus U$ is closed.

(c) If U is open, show that $E + U = \{e + u : e \in E, u \in U\}$ is open for every $E \subseteq \mathbb{R}$.

Solution. (a) (i) For any $x \in (0,1)$, we can choose $\delta_x = \min\{x, 1-x\}$ such that

$$(x - \delta_x, x + \delta_x) \subseteq [0,1],$$

therefore x is an interior point of $[0,1]$.

(ii) For any $r \in (0,\infty)$, both $(-r,r) \not\subseteq [0,1]$ and $(1-r, 1+r) \not\subseteq [0,1]$, so they are not interior point.

(b) By definition, the set $\mathbb{R} \setminus U$ is closed if and only if its complement $\mathbb{R} \setminus (\mathbb{R} \setminus U) = U$ is open.

(c) We note that

$$E + U = \bigcup_{e \in E} (e + U).$$

But the translation of an open set is still open, it is because

$$e + U = e + \bigsqcup (a_i, b_i) = \bigsqcup (e + a_i, e + b_i),$$

is a union of (disjoint) open intervals. Therefore $E + U$ is open as it is a union of open sets.

Notation: Here $\bigsqcup$ is no more than $\bigcup$ with the emphasis that the sets being "unioned" are pairwise disjoint.

Example 2. Study the openness and closedness of the following sets:

(a) $\{0\}$; (b) $\bigcup_{n\in\mathbb{Z}}(n-1,n)$; (c) $\mathbb{Z}$; (d) $\mathbb{Q}$; (e) Cantor Set C;

(f) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$; (g) $\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$; (h) $\emptyset$; (i) $\mathbb{R}$

Solution. (a) It is **closed** since $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$ is open.

(b) It is **open** since it is a union of open sets.

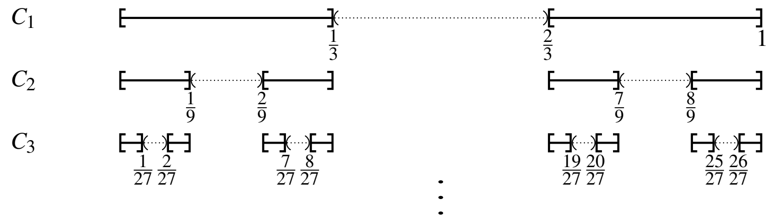
(c) It is **closed** since $\mathbb{R} \setminus \mathbb{Z} = \bigcup_{n\in\mathbb{Z}}(n-1,n)$ is open.

(d) It is **neither closed nor open**.

If $\mathbb{Q}$ is open, pick $q \in \mathbb{Q}$, then there is $r > 0$ s.t. $(q-r, q+r) \subseteq \mathbb{Q}$, but by density there is an $a \in \mathbb{R} \setminus \mathbb{Q}$, $a \in (q-r, q+r) \subseteq \mathbb{Q}$, a contradiction.

If $\mathbb{Q}$ is closed, then $\mathbb{R} \setminus \mathbb{Q}$ is open, again by density of $\mathbb{Q}$ this is impossible.

(e) It is **closed** since it is an intersection of closed sets C_n's inductively defined below:



(f) It is **not closed** since

$$\mathbb{R} \setminus \left\{\frac{1}{n} : n \in \mathbb{N}\right\} = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n}\right) \cup (1, \infty)$$

is not open as 0 is not an interior point.

(g) It is **closed** since

$$\mathbb{R} \setminus \left(\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}\right) = (-\infty, 0) \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n}\right) \cup (1, \infty)$$

is a union of open sets.

(h), (i) Both are **open and closed** by definition.

Exercise 1. Show that $A^\epsilon = \left\{x \in \mathbb{R} : \inf\{|x-a| : a \in A\} < \epsilon\right\}$ is open.

Example 3 (Sequential Closure). Show that the following are equivalent:

(a) The set $A \subseteq \mathbb{R}$ is closed.

(b) For every $x \in \mathbb{R}$ that is a limit of some sequence in A, $x \in A$.

Solution. (a) $\Rightarrow$ (b) Let x be a limit of some sequence in A, i.e., $x = \lim_{n\to\infty} a_n$, for some $a_n \in A$, we need to show $x \in A$. If not, i.e., $x \in \mathbb{R} \setminus A$, since A is closed, $\mathbb{R} \setminus A$ is open, so there is an $r > 0$ such that

$$(x - r, x + r) \subseteq \mathbb{R} \setminus A.$$

But $x \lim_{n\to\infty} a_n$, there must be an N such that $n > N \implies |x - a_n| < r$, i.e., $a_n \in (x - r, x + r)$, this a a contradiction as there will be (infinitely many) n such that $a_n \in (x - r, x + r) \subseteq \mathbb{R} \setminus A$ and $a_n \in A$.

(b) $\Leftarrow$ (a) To show A is closed, we try to show $\mathbb{R} \setminus A$ is open. Let $y \in \mathbb{R} \setminus A$, we hope there is an $r > 0$ such that $(y - r, y + r) \subseteq \mathbb{R} \setminus A$.

For the sake of contradiction let's suppose there is no such $r > 0$, i.e., let's assume for each $r > 0$,

$$(y - r, y + r) \not\subseteq \mathbb{R} \setminus A \iff (y - r, y + r) \cap A \neq \emptyset.$$

In particular, for each n we take $r = 1/n$, then there will be an $a_n \in A$ such that

$$a_n \in (y - 1/n, y + 1/n) \iff |y - a_n| < \frac{1}{n},$$

therefore

$$y = \lim_{n\to\infty} a_n.$$

By hypothesis $y \in A$, a contradiction to that $y \in \mathbb{R} \setminus A$ originally.

Extra Examples

Example 4. Let $K \subseteq \mathbb{R}$, $f : K \to \mathbb{R}$ be continuous. Show that

$$K \text{ is compact} \implies f(K) := \{f(x) : x \in K\} \text{ is compact.}$$

Solution. ***f(K)* is closed.** We use the sequential criterion in Example 3. Indeed, let $x \in \mathbb{R}$ be s.t. $x = \lim_{n\to\infty} f(k_n)$, for some $k_n \in K$. We try to show $x \in f(K)$. As $\{k_n\}$ is bounded, it has a convergent subsequence $\{k_{n_p}\}$ such that $k_{n_p} \to k$. Since K is closed, $k \in K$ by Example 3. Therefore $x = \lim_{p\to\infty} f(k_{n_p}) = f(k)$ by continuity, thus $x \in f(K)$.

***f(K)* is bounded.** We apply Supremum Limit Theorem to $\sup|f|(K)$. If $\sup|f|(K) = \infty$, then there is a sequence in $x_n \in K$ such that $|f(x_n)| \to \infty$. But $\{x_n\}$ is bounded, it has a convergent subsequence $\{x_{n_k}\}$ with $x_{n_k} \to k$ for some $k \in K$ (recall that since K is closed, k must be in K, c.f. Example 3), therefore by continuity we have

$$\mathbb{R} \ni |f(k)| = \lim_{k\to\infty} |f(x_{n_k})| = \infty,$$

a contradiction.

Exercise 2. Example 4 in particular shows that every continuous function defined on a compact set is bounded. Show the converse:

Let $K \subseteq \mathbb{R}$, show that if every continuous function defined on K is bounded, then K is compact.

Example 5. Let $K_1, K_2, \cdots \subseteq \mathbb{R}$ be a descending sequence of compact sets, i.e.,

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots.$$

Show that if $K_n \neq \emptyset$ for each n, then $\bigcap_{n=1}^{\infty} K_n$ is a **nonempty** compact set.

Solution. Let $K = \bigcap_{n=1}^{\infty} K_n$. K is of course compact since it is an intersection of closed sets and $K \subseteq K_1$. Now we show that $K \neq \emptyset$ by exhibiting an element in K.

For this, for each n we pick $x_n \in K_n$, as $\{x_n\}$ is bounded, it has a convergent subsequence $\{x_{n_k}\}$, $x_{n_k} \to x$. We show that $x \in K$. Indeed, for each fixed $p \in \mathbb{N}$, there is N such that

$$k > N \implies n_k > p \implies K_{n_k} \subseteq K_p.$$

So for $k > N$, $x_{n_k} \in K_p$ and thus $x = \lim_{k\to\infty} x_{n_k} \in K_p$ by Example 3. Since p is arbitrary, $x \in K := \bigcap_{n=1}^{\infty} K_n$.

Exercise 3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies two conditions:

(i) For each compact set K, $f(K)$ is compact.

(ii) For any descending sequence of compact sets $K_1 \supseteq K_2 \supseteq \cdots$,

$$f\left(\bigcap_{n=1}^{\infty} K_n\right) = \bigcap_{n=1}^{\infty} f(K_n).$$

Use the idea in the example above to prove that f is continuous.

Example 6 (Dini's Theorem). Let $f_n : [a,b] \to \mathbb{R}$ be continuous and $f_n \to f$ pointwise on $[a,b]$. Suppose also that:

(i) $f_n(x)$ is pointwise increasing.

(ii) $f(x)$ is continuous.

Show that $f_n \rightrightarrows f$ on $[a,b]$ by using the finite covering result: Theorem 9.

Solution. **Method 1 (Descending Compact Sets Technique).** We try to use Example 5. We know that for any fixed $\epsilon > 0$,

$$\bigcap_{n=1}^{\infty} \underbrace{\{x \in [a,b] : f - f_n \geq \epsilon\}}_{K_n} = \emptyset.$$

Since f and f_n are continuous, so is $f - f_n$, thus by Topological Continuity Theorem the set K_n is a closed subset of $[a,b]$, so K_n is compact. By Example 5 there is an N such that $K_N = \emptyset$ since $\{K_n\}$ is descending. Now $n \geq N \implies K_n = \emptyset$, thus for any $x \in [a,b]$ and for any $n \geq N$, $f(x) - f_n(x) < \epsilon$ (iff $x \notin K_n = \emptyset$), which is the definition of uniform convergence.

Method 2 (Open Covering Technique). Fix $\epsilon > 0$, for every $x \in [a,b]$, there is an $n \in \mathbb{N}$ such that

$$f(x) - f_n(x) < \epsilon,$$

thus $x \in \bigcup_{n=1}^{\infty} (f - f_n)^{-1}(-\infty, \epsilon)$. Note that this is true for each $x \in [a,b]$, we have

$$[a,b] \subseteq \bigcup_{n=1}^{\infty} (f - f_n)^{-1}(-\infty, \epsilon). \tag{$*$}$$

Note that it becomes straightforward to create an open cover of $[a,b]$. By hypothesis for each n both f and f_n are continuous, so by Topological Continuity Theorem there is an open set $U_n \subseteq \mathbb{R}$ such that

$$(f - f_n)^{-1}(-\infty, \epsilon) = U_n \cap [a,b].$$

It follows from $(*)$ that $[a,b] \subseteq \bigcup_{n=1}^{\infty} U_n$. Since $[a,b]$ is **compact**, by Theorem 9 the open cover $\{U_n\}$ of $[a,b]$ can be thinned into an **finite subcover**. We may also include redundant sets in $\{U_n\}$ to assume $[a,b] \subseteq \bigcup_{n=1}^{N} U_n$ for some $N \in \mathbb{N}$. Thus

$$[a,b] = \bigcup_{n=1}^{N} (U_n \cap [a,b]) = \bigcup_{n=1}^{N} (f - f_n)^{-1}(-\infty, \epsilon) = (f - f_N)^{-1}(-\infty, \epsilon).$$

The last equality follow from the fact that $\left\{(f - f_n)^{-1}(-\infty, \epsilon)\right\}_{n=1}^{\infty}$ is ascending.

Now for every $x \in [a,b]$, for every $n \geq N$, $f(x) - f_n(x) \leq f(x) - f_N(x) < \epsilon$, thus $f_n \rightrightarrows f$ on $[a,b]$.