(Skipped) Differentiation (Part II): Taylor Series

We need to know

- how is the Taylor series expansion used in constructing various estimates.

Key definitions and results

Theorem 1 (Taylor's). Let $f(x)$ be a continuous on $[a,b]$ and n-times differentiable on (a,b). For every $x \in [a,b]$ and $\alpha \in (a,b)$, there is c between x and α such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k + \frac{f^{(n)}(c)}{n!}(x-\alpha)^n.$$

We call $R_n(x) := \frac{f^{(n)}(c)}{n!}(x-\alpha)^n$ the **Lagrange form** of the remainder.

Remark. One may doubt whether we should impose $x \in [a,b]$ or $x \in (a,b)$ in the Taylor's Theorem. If one examines the proof carefully the Lagrange form of the remainder holds whenever $f(x)$ has nth order derivative **between** x and α (in order to apply Generalized Mean-Value Theorem). Thus $x \in [a,b]$ is a **safe** requirement!

Remark. Suppose $f(x)$ is n-times differentiable on (a,b), $x \in [a,b]$ and h small enough such that $x+h \in (a,b)$. The following form of Taylor series will be **very useful**!

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \cdots + \frac{f^{(n-1)}(x)}{(n-1)!}h^{n-1} + \frac{f^{(n)}(x+\theta h)}{n!}h^n$$

for some $\theta \in (0,1)$. The is obtained by replacing x by $x+h$ and α by x in Taylor's Theorem.

Remark. The remainder term of Taylor series has two more different forms, one is for those $f^{(n)}(x)$ integrable (the **integral form**) and one is for those $f^{(n)}(x)$ continuous (the **Cauchy form**). They can be found in lecture notes and we will not use them in this tutorial note.

Example 1 (nth Order Approximation). Let $f(x)$ be n-times differentiable on (a,b) and $x_0 \in (a,b)$. Let

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

denote the n-th Taylor series of $f(x)$ at x_0. Show that this is an ***n-th order approximation*** of $f(x)$ in the sense that

$$\lim_{x\to x_0} \frac{f(x)-T_n(x)}{(x-x_0)^n} = 0$$

by using the Generalized Mean-Value Theorem.

Remark. This result is usually written as

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + o((x-x_0)^k) \quad \text{as } x \to x_0.$$

Sol In fact this is a very well-known theorem and can be found very powerful in computation of limits. Let's denote $R_n = f - T_n$, then we observe that

$$R_n(x_0) = R_n'(x_0) = \cdots = R_n^{(n)}(x_0) = 0.$$

Therefore for every $x \in (a,b)$, by Generalized Mean-Value Theorem we have

$$\begin{aligned}
\frac{R_n(x)}{(x-x_0)^n} &= \frac{R_n(x)-R_n(x_0)}{(x-x_0)^n-(x_0-x_0)^n} \\
&= \frac{R_n'(c_1)}{n(c_1-x_0)^{n-1}} && (\exists c_1 \text{ btw } x, x_0) \\
&= \frac{R_n'(c_1)-R_n'(x_0)}{n((c_1-x_0)^{n-1}-(x_0-x_0)^{n-1})} \\
&= \frac{R_n''(c_2)}{n(n-1)(c_2-x_0)^{n-2}} && (\exists c_2 \text{ btw } c_1, x_0) \\
&= \cdots && \vdots \\
&= \frac{R_n^{(n-1)}(c_{n-1})}{n(n-1)\cdots 2(c_{n-1}-x_0)} && (\exists c_{n-1} \text{ btw } c_{n-2}, x_0) \\
&= \frac{1}{n!}\cdot\frac{R_n^{(n-1)}(c_{n-1})-R_n^{(n-1)}(x_0)}{c_{n-1}-x_0}. && (*)
\end{aligned}$$

Now c_{n-1} depends on x, and we note that if $x < x_0$, then

$$x < c_1 < c_2 < \cdots < c_{n-1} < x_0;$$

or if $x > x_0$, then

$$x > c_1 > c_2 > \cdots > c_{n-1} > x_0.$$

In either case, we have when $x \to x_0$, $c_{n-1} \to x_0$, therefore by the definition of derivative,

$$\lim_{x\to x_0} \frac{R_n(x)}{(x-x_0)^n} = \frac{1}{n!} R_n^{(n)}(x_0) = 0. \quad ∎$$

Remark. With the same hypothesis in this example we can use Generalized Mean-Value Theorem once more in $(*)$ to obtain

$$\frac{f(x)-T_n(x)}{(x-x_0)^n} = \frac{1}{n!} R_n^{(n)}(c_n)$$

for some c_n between x and x_0. But this does not imply our desired conclusion because $R_n^{(n)}$ is not necessarily continuous at x_0.

Remark. From the proof we can refine the hypothesis of the example as follows. Note that we just require $R_n', R_n'', \ldots, R_n^{(n-1)}$ exist everywhere near x_0 in order to apply Generalized Mean-Value Theorem $n-1$ times, and we require $R_n^{(n)}(x_0)$ exists at at the last step.

In other words, since $R_n = f - T_n$, where T_n is a polynomial (hence infinitely differentiable everywhere), the **minimal hypothesis** that we need to impose is:

- $f : (a,b) \to \mathbb{R}$ is such that $f', f'', \ldots, f^{(n-1)}$ exist everywhere near x_0.
- $f^{(n)}(x_0)$ exists.

Combing these two, we have the same conclusion in this example.

Example 2. Suppose $f(x)$ is thrice (i.e., 3-times) differentiable on $(-1,1)$ such that

$$f(-1) = 0, \quad f(1) = 1 \quad f'(0) = 0.$$

Prove that $f^{(3)}(x_0) \geq 3$ for some $x_0 \in (-1,1)$.

Sol See lecture notes. ∎

Example 3. Suppose $f(x)$ is differentiable on (a,∞) and $|f''(x)|$ is bounded on (a,∞). Prove that

$$\lim_{x\to\infty} f(x) = 0 \implies \lim_{x\to\infty} f'(x) = 0.$$

Sol Since f'' is bounded on (a,∞), by definition there is an M such that $|f''(x)| \le M$ for every $x > a$.

We will use the following form of Taylor series. Let $h > 0$, then

$$f(x+h) = f(x) + f'(x)h + f''(x+\theta h)\frac{h^2}{2}$$

for some $\theta \in (0,1)$.

By the definition of $\lim_{x\to\infty} f(x) = 0$, for every $\epsilon > 0$, there is $b > 0$ such that

$$y > b \implies |f(y)| < \epsilon.$$

Therefore for every $h > 0, x > b$, we also have $x + h > b$, hence

$$\left| f'(x)h + f''(x+\theta h)\frac{h^2}{2} \right| = |f(x+h) - f(x)| \le |f(x+h)| + |f(x)| < 2\epsilon.$$

On the other hand,

$$\left| f'(x)h + f''(x+\theta h)\frac{h^2}{2} \right| \ge |f'(x)|h - |f''(x+\theta h)|\frac{h^2}{2} \ge |f'(x)|h - M\cdot\frac{h^2}{2},$$

so for every $h > 0$, combining the above estimates we have

$$x > b \implies |f'(x)| < \frac{2\epsilon + m\frac{h^2}{2}}{h} = 2\epsilon\frac{1}{h} + \frac{M}{2}h. \qquad (*)$$

Since $(*)$ holds for every $h > 0$, we may take $h = \sqrt{\epsilon}$ and conclude that

- for any $\epsilon > 0$,
- there is a $b > 0$ such that,
- $x > b \implies |f'(x)| < \left(2 + \frac{M}{2}\right)\sqrt{\epsilon}$.

Since M is a constant, for every $\varepsilon > 0$, we may choose $\epsilon = \varepsilon^2 \Big/ \left(2 + \frac{M}{2}\right)^2$ at the beginning such that there is $b > 0$,

$$x > b \implies |f'(x)| < (2 + \frac{M}{2})\sqrt{\frac{\varepsilon^2}{(2+\frac{M}{2})^2}} = \varepsilon,$$

thus $\lim_{x\to\infty} |f'(x)| = 0$. ∎

Remark. $(*)$ holds for every $h > 0$, we may take $h = 2\sqrt{\epsilon/M}$ to attain the minimum of the rightmost quantity, then for every $\epsilon > 0$, there is $b > 0$,

$$x > b \implies |f'(x)| < 2\sqrt{M\epsilon}.$$

Example 4. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable such that $M_k := \sup_{x\in\mathbb{R}} |f^{(k)}(x)| < \infty$ for $k = 0, 1, 2$. Show that

$$M_1 \le \sqrt{2M_0M_2}.$$

Sol For every $x \in \mathbb{R}$ and $h \in \mathbb{R}$, we still use the following form of Taylor series:

$$f(x+h) = f(x) + f'(x)h + f''(x+\theta h)\frac{h^2}{2}$$

and

$$f(x-h) = f(x) - f'(x)h + f''(x+\theta' h)\frac{h^2}{2}.$$

By subtracting them, we have

$$f(x+h) - f(x-h) = 2f'(x)h + \left(f''(x+\theta h) - f''(x+\theta' h)\right)\frac{h^2}{2}.$$

Using the definition of M_0, M_1 and M_2, we have

$$0 = \left| \begin{array}{c} f(x-h) - f(x+h) + 2f'(x)h \\ + \\ \left(f''(x+\theta h) - f''(x+\theta' h)\right)\frac{h^2}{2} \end{array} \right| \le 2M_0 + 2M_1 h + M_2 h^2.$$

Since this holds for every $h \in \mathbb{R}$ (as the domain of f is $\mathbb{R}$), it follows that the degree two polynomial

$$M_2h^2 + 2M_1h + 2M_0$$

either has only one solution or has no solution, i.e.,

$$\Delta = 4M_1^2 - 4M_2 \cdot 2M_0 \le 0 \iff M_1 \le \sqrt{2M_0M_2}. \quad ∎$$

Remark. We give a generalization in Exercise 5. Note that if $f(x)$ is just differentiable on (a, ∞) for some $a \in \mathbb{R}$, then we have a weaker inequality $J_1 \le 2\sqrt{J_0J_2}$, where $J_k = \sup_{x\in(a,\infty)} |f^{(k)}(x)| < \infty$, $k = 0, 1, 2$, c.f. Rudin p.115 or Presentation Exercise 78.

Exercises

1. Let $f(x)$ be defined near a such that $f(a) \ne 0$ and $f'(a)$ exists, show that
$$\lim_{n\to\infty} \left(\frac{f(a+1/n)}{f(a)}\right)^n = e^{f'(a)/f(a)}.$$

2. Suppose $f(x)$ has second order derivative near 0 (i.e., on $(-\delta, \delta)$, $\exists\delta > 0$). Let
$$\lim_{x\to 0} \left(1 + x + \frac{f(x)}{x}\right)^{1/x} = e^{\lambda}.$$
Find $f(0), f'(0)$ and $f''(0)$, and then find $\lim_{x\to 0} \left(1 + \frac{f(x)}{x}\right)^{1/x}$.

3. **(2007 Spring)** Let $f(x)$ be thrice differentiable on $\mathbb{R}$. If $f(x)$ and $f'''(x)$ are bounded on $\mathbb{R}$, show that $f'(x)$ and $f''(x)$ are also bounded on $\mathbb{R}$.

4. Let $f(x)$ be twice differentiable on $[a, b]$ and $f'(a) = f'(b) = 0$, prove that there is a $c \in (a, b)$ such that
$$|f''(c)| \ge \frac{4}{(b-a)^2}|f(b) - f(a)|.$$

5. Let $f(x)$ be p-times differentiable on $\mathbb{R}$ and $M_j = \sup_{x\in\mathbb{R}} |f^{(j)}(x)| < \infty$ for $j = 0, 1, 2, \ldots, p$, where $p \ge 2$. Show that for every $1 \le k \le p-1$,
$$M_k \le 2^{k(p-k)/2} M_0^{1-k/p} M_p^{k/p}.$$

6. **(2007 Spring)** Let $f : [0, 1] \to \mathbb{R}$ be continuous and $f(0) = f(1)$. If $f(x)$ is twice differentiable on $(0, 1)$, and there is $M > 0$ such that $|f''(x)| \le M$ for all $x \in (0, 1)$, then prove that $|f'(x)| \le \frac{1}{2}M$ for all $x \in (0, 1)$.

7. **(2010 Spring)** Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable such that for every $x \in [0, 1]$, $|f''(x)| \le 2010$. If there is $c \in (0, 1)$ such that $f(c) > f(0)$ and $f(c) > f(1)$, prove that
$$|f'(0)| + |f'(1)| \le 2010.$$

8. **(Putnam 2007)** Suppose $f : [0, 1] \to \mathbb{R}$ has continuous derivative on $[0, 1]$ (implicitly, $f'_+(0), f'_-(0)$ exist) and $\int_0^1 f(x)\,dx = 0$. Prove that for every $\alpha \in (0, 1)$,
$$\left|\int_0^{\alpha} f(x)\,dx\right| \le \frac{1}{8} \max_{0\le x\le 1} |f'(x)|.$$