Math2033 Mathematical Analysis (Spring 2013-2014) **Tutorial Note 6**

Differentiation (Part I): L'Hôpital's Rule & Mean-Value Theorem

We need to know

- the *integrating factor technique* when we try to apply Mean-Value Theorem.
- the power of more general form of L'Hôpital's rule—the $\frac{*}{\infty}$ version.

Key definitions and results

Definition 1 (Differentiability). Let I be an open interval. A function $f : I \to \mathbb{R}$ is **differentiable** at $x_0 \in I$ if the limit

$$f'(x_0) := \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

exists. $f(x)$ is said to be **differentiable** if it is differentiable at every point of I.

Theorem 2 (L'Hôpital's Rule, $\frac{0}{0}$ Version). Let $f(x), g(x)$ be differeitable on (a,b) and both $g(x), g'(x) \neq 0$ for all $x \in (a,b)$, where $-\infty \le a < b \le +\infty$. Suppose that

(a) $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, where $-\infty \le L \le +\infty$, and

(b) $\lim_{x\to a^+} f(x) = 0 = \lim_{x\to a^+} g(x)$,

then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$. (Similarly, the rule is also true if $x \to b^-$.)

Theorem 3 (L'Hôpital's Rule, $\frac{*}{\infty}$ Version). Let $f(x), g(x)$ be differeitable on (a,b) and both $g(x), g'(x) \neq 0$ for all $x \in (a,b)$, where $-\infty \le a < b \le +\infty$. Suppose that

(a) $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, where $-\infty \le L \le +\infty$, and

(b) $\lim_{x\to a^+} g(x) = +\infty$,

then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$. (Similarly, the rule is also true if $x \to b^-$.)

Remark. In Math1014, Calculus II, we have already learnt the $\frac{\infty}{\infty}$ version of L'Hôpital's Rule. The $\frac{*}{\infty}$ version we have stated here is much better in the sense that the numerator $f(x)$ needs not have ∞ has its "limit" (i.e., it is allowed to behave arbitrarily).

Theorem 4 (Inverse Function). If $f(x)$ is continuous, injective on (a,b) and $f'(x_0) \neq 0$ for some $x_0 \in (a,b)$, then f^{-1} is differentiable at $y_0 = f(x_0)$, moreover, $(f^{-1})'(y_0) = 1/f'(x_0)$.

Theorem 5 (Local Extreme). Let $f(x)$ be differentiable on (a,b). If $x_0 \in (a,b)$ satisfies

$$f(x_0) = \min f\big((a,b)\big) \quad \text{or} \quad f(x_0) = \max f\big((a,b)\big),$$

then $f'(x_0) = 0$.

Remark. The Local Extreme Theorem **only** applys for local maximum and local minimum on **open intervals**.

Theorem 6 (Mean-Value). Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b), then there is an $x_0 \in (a,b)$ such that

$$f(b) - f(a) = f'(x_0)(b-a).$$

Theorem 7 (Generalized Mean-Value). Let $f(x), g(x)$ be continuous on $[a,b]$ and differentiable on (a,b), then there is a $\zeta \in (a,b)$ such that

$$g'(\zeta)(f(b)-f(a)) = f'(\zeta)(g(b)-g(a)).$$

Remark (Integrating Factor). In solving ODE we are introduced the following useful technique: Given $f(x), p(x)$ *nice* and the following expression

$$f'(x) + p(x)f(x),$$

we can multiply this by an **integrating factor**—$e^{P(x)}$, where $P(x)$ "=" $\int p(x)\,dx$, so as to rewrite the above as

$$f'(x)+p(x)f(x) = \boldsymbol{e}^{-P(x)}[\boldsymbol{e}^{P(x)}(f'(x)+p(x)f(x))] = \boxed{e^{-P(x)}(e^{P(x)}f(x))'.}$$

This will be helpful when we try to seek for a function to which we apply Mean-Value Theorem. We illustrate this in Example 3. Similar situation also happens in Exercise 10 (harder!) when we try to write $P - rP'$ as (sth)$'$.

Example 1. Suppose that $f(x) \geq 0$ and differentiable on $(0,1)$. Show that if $f(x) = 0$ for at least two $x \in (0,1)$, then $f'''(c) = 0$ for some $c \in (0,1)$.

Sol Let $a, b \in (0,1)$ be such that $f(a) = f(b) = 0$. By Mean-Value Theorem we directly have

$$f(a) - f(b) = 0 = f'(d)(b-a), \quad \text{for some } d \in (a,b),$$

i.e., $f'(d) = 0$ for some $d \in (a,b)$.

Since we are given that $f(x) \geq 0$, thus $f(a) = 0$ and $f(b) = 0$ are local (in fact, global) minimum, therefore

$$f'(a) = 0 \quad \text{and} \quad f'(b) = 0.$$

Now $a < d < b$ and $f'(a) = f'(d) = f'(b) = 0$, by applying Mean-Value Theorem on $[a,d]$ and $[d,b]$ respectively to $f'(x)$, we have $f''(c_1) = 0$ and $f''(c_2) = 0$ for some $c_1 \in (a,d)$ and $c_2 \in (d,b)$.

Since $f''(c_1) = f''(c_2) = 0$, another application of Mean-Value Theorem gives $f'''(c) = 0$ for some $c \in (c_1, c_2) \subseteq (a,b) \subseteq (0,1)$. ∎

Example 2. Let $f(x)$ be twice differentiable on $(0,\infty)$ such that

$$\lim_{x\to\infty} f(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} f''(x) = L$$

for some $L \in \mathbb{R}$. Show that $\lim_{x\to\infty} f'(x) = L$ and then show that $L = 0$.

Sol This example can be solved by applying L'Hôpital's Rule several times.

Note that we have

$$\begin{aligned}\lim_{x\to\infty}(f(x) - f'(x)) &= \lim_{x\to\infty} \frac{e^x(f(x) - f'(x))}{e^x} \\ &\overset{\text{if RHS } \exists}{=\!=\!=} \lim_{x\to\infty} \frac{e^x(f(x) - f'(x) + f'(x) - f''(x))}{e^x} \\ &= \lim_{x\to\infty}(f(x) - f''(x)) \\ &= 0 - L.\end{aligned}$$

Since $\lim_{x\to\infty} f(x)$ also exists, thus $\ell = \lim_{x\to\infty} f'(x)$ exists and the above gives

$$0 - \ell = -L \implies \ell = L.$$

By applying the same trick once more, we have

$$0 = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^x f(x)}{e^x} \overset{\text{if RHS } \exists}{=\!=\!=} \lim_{x\to\infty} \frac{e^x(f(x) + f'(x))}{e^x} = 0 + L. \quad ∎$$

Remark. In fact even more is true. If $f : (0,\infty) \to \mathbb{R}$ is twice differentiable, $\lim_{x\to\infty} f(x) = 0$ with $f''(x)$ bounded, then $\lim_{x\to\infty} f'(x) = 0$. This follows from careful use of Taylor expansion and will be done in the next tutorial note.

Example 3. Let $f(x)$ be continuous on $[a,b]$, differentiable on (a,b) and $f(a) = f(b) = 0$. Prove that for each $\lambda \in \mathbb{R}$, there is $c \in (a,b)$ such that

$$f'(c) = \lambda f(c).$$

Sol Note that $f'(c) - \lambda f(c) = 0$

iff $f'(x) - \lambda f(x) = 0$ at $x = c$
iff $e^{-\lambda x}(f'(x) - \lambda f(x)) = 0$ at $x = c$
iff $(e^{-\lambda x} f(x))' = 0$ at $x = c$.

Let $g(x) = e^{-\lambda x} f(x)$, then we need to show there is $c \in (a,b)$ such that $g'(c) = 0$.

But this follows directly from Mean-Value Theorem since $g(a) = g(b) = 0$. ∎

Example 4 (Intermediate Value Property). Let $f : (a,b) \to \mathbb{R}$ be differentiable. Let's suppose also that

- $u, v \in (a,b)$ satisfy $f'(u) \neq f'(v)$;
- $m \neq f'(u), f'(v)$ be between $f'(u), f'(v)$.

Show that there is a $c \in (u,v)$ such that $f'(c) = m$.

Sol First we assume $f'(u) < f'(v)$. Let $m \in (f'(u), f'(v))$, we need find $c \in (u,v)$ such that

$$f'(c) = m \iff (f(x) - mx)'\Big|_{x=c} = 0.$$

Let's define $g(x) = f(x) - mx$.

By hypothesis, $g'(u) = f'(u) - m < 0$ and $g'(v) = f'(v) - m > 0$. Also, by Extreme Value Theorem there is $c \in [u,v]$ such that $g(c) = \min g([u,v])$.

We expect $g'(c) = 0$, for this, we need to show $c \neq u, v$ in other to apply Local Extreme Theorem (which only works for local min or max over an open interval).

To prove $c \neq u$, we try to show $g(u)$ cannot be a global min on $[u,v]$ (therefore $g(c) \neq g(u) \implies c \neq u$). Since $g'(u) < 0$, which means that there is $x > u$ such that $g(x) < g(u)$ (otherwise if $g(x) \geq g(u)$ for all $x \geq u$, then this implies $g'(u) \geq 0$, a contradiction). We conclude $c \neq u$.

Similarly, since $g'(v) > 0$, $g(v)$ cannot be a global min on $[u,v]$. Thus $c \neq v$.

Now $g(c)$ is a global min on $[u,v]$, it is a local min on (u,v) (as it is shown that $c \neq u, v$), thus $g'(c) = 0$, as desired.

Finally we handle the case that $f'(u) > f'(v)$. Note that we can return to the previous situation: $-f(x)$ is differentiable on (a,b) and $(-f)'(u) < (-f)'(v)$.

Applying $-f(x)$ to what we have just proved, we have for every $-m \in ((-f)'(u), (-f)'(v))$, there is $c \in (u,v)$ such that $(-f)'(c) = -m$, thus $f'(c) = m$. ∎

Remark. If $f'(x)$ is continuous, then the conclusion holds immediately by Intermediate Value Theorem. This example asserts that "Intermediate Value Property" is still possible even if $f'(x)$ is not continuous.

Exercises

1. A function $g(x)$ on (a,b) is said to have a **simple discontinuity** at x if

$$g(x^-) := \lim_{t\to x^-} g(t) \quad \text{and} \quad g(x^+) := \lim_{t\to x^+} g(t)$$

exist but $g(x^-) \neq g(x^+)$. Suppose that $f(x)$ is differentiable on (a,b).

(a) Show that $f'(x)$ cannot have any simple discontinuities in (a,b).

Hint: Recall that $f'(x)$ possesses Intermediate Value Property mentioned in Example 4.

(b) If $\lim_{x\to c} f'(x)$ exists for some $c \in (a,b)$, show that $f'(x)$ is continuous at c.

2. Let $f(x), g(x)$ be differentiable on (a,b) such that

$$f(x)g'(x) - f'(x)g(x) \neq 0 \quad \text{for every } x \in (a,b).$$

If there are x_0, x_1 such that $a < x_0 < x_1 < b$ and $f(x_0) = f(x_1) = 0$, then prove that there exists x_0 such that $f''(x_0) = 0$.

3. Let $f(x)$ be a differentiable function on $(0,\infty)$. Suppose $f(x)$ is bounded, i.e., there is an M such that $|f(x)| \leq M$ for all $x \in (0,\infty)$. Show that there is a sequence of numbers $\{x_n\}$, $x_n \to \infty$, such that $f'(x_n) \to 0$ as $n \to \infty$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$\left| f(x) - \sin(x^2) \right| \leq \frac{1}{4} \quad \text{for every } x \in \mathbb{R}.$$

Prove that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} f'(x_n) = +\infty$.

5. Let $g(x)$ be a differentiable on $\mathbb{R}$ and $g'(x)$ bounded on $\mathbb{R}$. Show that if $\delta > 0$ is small enough, then $f(x)$ on $\mathbb{R}$ given by

$$f(x) = x + \delta g(x)$$

is bijective.

6. Let $f(x), g(x), h(x)$ be continuous on $[a,b]$ and differentiable on (a,b), show that there is $c \in (a,b)$ such that

$$\det \begin{bmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(c) & g'(c) & h'(c) \end{bmatrix} = 0.$$

Note that the Generalized Mean-Value Theorem is the case that $h \equiv 1$.

7. Let $f : \mathbb{R} \to (0,\infty)$ be a differentiable function such that $f'(x) = f(f(x))$ for all $x \in \mathbb{R}$. Show that there is no such function.

8. **(2007 Spring)** Let $f(x)$ be continuous on $[0,1]$, differentiable on $(0,1)$, $f(0) = 0$ and $f(1) = 1$. Let $k_1, k_2, \ldots, k_n > 0$ be such that $k_1 + k_2 + \cdots + k_n = 1$. Prove that there are pairwise distinct $t_1, t_2, \ldots, t_n \in (0,1)$ such that

$$\frac{k_1}{f'(t_1)} + \frac{k_2}{f'(t_2)} + \cdots + \frac{k_n}{f'(t_n)} = 1.$$

9. Let $f(x)$ be defined on $(-\epsilon, \epsilon)$ and continuous at $x = 0$. If

$$\lim_{x\to 0} \frac{f(2x) - f(x)}{x} = m,$$

prove that $f(x)$ is differentiable at 0, moreover, $f'(0) = m$.

10. Let $P(x)$ be a nonconstant polynomial with real coefficients and only has real roots. Prove that for every $r \in \mathbb{R}$, the polynomial

$$Q_r(x) := P(x) - rP'(x)$$

only has real roots as well.

11. Suppose $f(x)$ is continuous on $[a,b]$, differentiable on (a,b), $f(a) = 0$ and $0 \leq f'(x) \leq 1$, prove that

$$\left(\int_a^b f(x)\, dx \right)^2 \geq \int_a^b f(x)^3\, dx.$$