Consequence of Uniform Convergence

Key Definitions and Results

Theorem 1 (Continuity Theorem for Uniform Convergence).

- **Sequence Version.** Let $f_n : E \to \mathbb{R}$ be a sequence of functions such that:

  (i) $f_n \rightrightarrows f$ on E.

  (ii) For each n, $\lim_{x\to c} f(x)$ exists.

  Then f also has limit at c: $\lim\limits_{x\to c}\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty}\lim\limits_{x\to c} f_n(x)$.

- **Series Version.** Let $f_n : E \to \mathbb{R}$ be a sequence of functions such that:

  (i) $\sum_{n=1}^{\infty} f_n$ converges uniformly on E.

  (ii) For each n, $\lim_{x\to c} f_n(x)$ exists.

  Then $\sum_{n=1}^{\infty} f_n$ also has limit at c: $\lim\limits_{x\to c}\sum\limits_{n=1}^{\infty} f_n(x) = \sum\limits_{n=1}^{\infty}\lim\limits_{x\to c} f_n(x)$.

Theorem 2 (Integration Theorem for Uniform Convergence).

- **Sequence Version.** Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of Riemann integrable functions such that $f_n \rightrightarrows f$ on $[a,b]$, then f is also Riemann integrable on $[a,b]$, moreover,
$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f\,dx.$$

- **Series Version.** Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of Riemann integrable functions. If $\sum_{n=1}^{\infty} f_n$ converges **uniformly**, then $\sum_{n=1}^{\infty} f_n$ is Riemann integrable on $[a,b]$, moreover,
$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty}\int_a^b f_n(x)\,dx.$$

Theorem 3 (Differentiation).

- **Sequence Version.** Let $f_1, f_2, \cdots : (a,b) \to \mathbb{R}$ be differentiable functions such that:

  (i) $f_n(x_0)$ converges for some $x_0 \in (a,b)$.

  (ii) $f_n'(x)$ converges uniformly on (a,b).

  Then f_n converges uniformly on (a,b) and $\left(\lim\limits_{n\to\infty} f_n(x)\right)' = \lim\limits_{n\to\infty} f_n'(x)$.

- **Series Version.** Let $f_1, f_2, \cdots : (a,b) \to \mathbb{R}$ be differentiable functions such that:

  (i) $\sum_{n=1}^{\infty} f_n(x_0)$ converges for some $x_0 \in (a,b)$.

  (ii) $\sum_{n=1}^{\infty} f_n'$ converges uniformly on (a,b).

  Then $\sum_{n=1}^{\infty} f_n$ converges uniformly on (a,b) and $\left(\sum\limits_{n=1}^{\infty} f_n(x)\right)' = \sum\limits_{n=1}^{\infty} f_n'(x)$.

Theorem 4 (Differentiation of Power Series). If $\sum\limits_{n=0}^{\infty} a_n(x-c)^n$ converges pointwise on (a,b), then it is differentiable on (a,b) with
$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n(x-c)^n\right) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}.$$

Remark. The Differentiation Theorem of Power Series states that whenever a power series converges on an open interval I, it is **infinitely differentiable on I**.

Theorem 5 (Abel's Limit). If $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ converges pointwise on a closed and bounded interval $[u,v]$, then the series converges uniformly on $[u,v]$.

Theorem 6. We have the following properties for uniform convergence:

- **Bounded Multiplier Property.** Suppose f is bounded on E and $g_n \rightrightarrows g$ on E, then $fg_n \rightrightarrows fg$ on E.

- **Substitution Property.** Suppose $g_n \rightrightarrows g$ on E and $f : F \to E$ is any function, then $g_n \circ f \rightrightarrows g \circ f$ on E.

Theorem 7 (Dini). Let $f_n : [a,b] \to \mathbb{R}$ be continuous and $f_n \to f$ pointwise on $[a,b]$. Suppose also that:

(i) $f_n(x)$ is pointwise increasing.

(ii) $f(x)$ is continuous.

Then f_n converges to f uniformly on $[a,b]$.

Example 1. Reconsider Example 6 in tutorial note 5. Show that the series

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$$

does not converge uniformly on $(0,1)$ by using the continuity theorem for uniform convergence.

Solution. For any $x \neq 0$ the series converges pointwise to

$$f(x) = x^2 \sum_{n=0}^{\infty} \frac{1}{(1+x^2)^n} = x^2 \cdot \frac{1+x^2}{x^2} = 1 + x^2.$$

If the convergence were uniform on $(0,1)$, then it would be uniform also on $[0,1]$ (easily seen by uniform Cauchy criterion, see the complete statement in the following "fact"), therefore f would be continuous on $[0,1]$ by Continuity Theorem. However,

$$f(x) = \begin{cases} 0, & x = 0, \\ 1+x^2, & x \in (0,1]. \end{cases}$$

Therefore the contrapositive of Continuity Theorem tells us

$$f \text{ not continuous} \implies \text{the convergence is not uniform.}$$

Fact. *Suppose that $f_1, f_2, \cdots : [a,b] \to \mathbb{R}$ is a sequence of continuous functions such that it converges uniformly on (a,b), then it also converges uniformly on $[a,b]$.*

Proof. If f_n converges uniformly on (a,b), then it is uniformly Cauchy on (a,b), thus for every $\epsilon > 0$, there is an N such that

$$m,n > N \implies \|f_m - f_n\|_{(a,b)} < \epsilon.$$

However, if $\|f_m - f_n\|_{(a,b)} < \epsilon$, then $|f_m(x) - f_n(x)| < \epsilon$ for every $x \in (a,b)$, by taking limit $x \to a^+$ and $x \to b^-$ respectively we have $|f_m - f_n| \le \epsilon$ on $[a,b]$, thus $\|f_m - f_n\|_{[a,b]} \le \epsilon$. We conclude that

$$m,n > N \implies \|f_m - f_n\|_{[a,b]} \le \epsilon,$$

so $\{f_n\}$ converges uniformly on $[a,b]$.

Example 2.

(a) Show that the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} \sin \frac{x}{k}$ converges uniformly on $(-1,1)$.

(b) How about the uniform convergence of $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} \cos \frac{x}{k}$ on $(-1,1)$?

Solution. (a) Since

- $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} \left(\sin \frac{x}{k}\right)' = \sum_{k=1}^{\infty} \frac{1}{k^2} \cos \frac{x}{k}$ converges uniformly on $(-1,1)$ by M-test.
- The series $\sum_{k=1}^{\infty} \frac{1}{k} \sin \frac{0}{k}$ converges with $0 \in (-1,1)$.

By the Differentiation Theorem the series converges uniformly.

(b) Although

$$\sum_{k=1}^{\infty} \frac{1}{k} \left(\cos \frac{x}{k}\right)' = -\sum_{k=1}^{\infty} \frac{1}{k^2} \sin \frac{x}{k}$$

converges uniformly on $(-1,1)$, we cannot apply the Differentiation Theorem yet because we still need to find a point in $(-1,1)$ at which the series converges.

Actually such point does not exist: for every $x \in \mathbb{R}$,

$$\sum_{k=1}^{\infty} \frac{1}{k} \cos \frac{x}{k} = \infty.$$

Exercise 1 (Math2043 2008 Final). Show that $\displaystyle\lim_{x\to 0} \frac{1}{x} \left(\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+x^2} \right)'$ exists and compute it. You may need to know how to compute Fourier series.

Exercise 2 (Math2043 2006 Final). Let $a_n > 0$, $r = \overline{\lim} \dfrac{\ln a_n}{\ln n}$ and $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n^x}$.

(a) Prove the for any $r' > r$, we have $a_n < n^{r'}$ for sufficiently big n.

(b) Prove the for any $R > r+1$, the series uniformly converges on $[R, \infty)$.

(c) Prove the $f(x)$ has derivatives of any order on $(r+1, \infty)$.

(d) Prove that the series diverges on $(-\infty, r)$. On the other hand, show that the series may converge for some $x < r+1$ by constructing an example.

Example 3 (Practice Exercise #59). Show that

$$\int_0^1 \frac{1-\cos x^2}{x^4}\,dx = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)!(4k-3)}.$$

Solution. Note that LHS is an improper integral, we must switch to proper integral in order to apply Integration Theorem. For this, we use the following result:

Theorem. *If $f:(a,b]\to\mathbb{R}$ is locally integrable and bounded near a, then f is improper integrable, moreover,*

$$\int_a^b f\,dx = \int_a^b \tilde{f}\,dx,$$

for any $\tilde{f}:[a,b]\to\mathbb{R}$ such that $f=\tilde{f}$ on $(a,b]$, i.e., for any function $\tilde{f}$ that extends f to $[a,b]$.

Now for any $x \neq 0$ we have

$$\frac{1-\cos x^2}{x^4} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^{4k-4}}{(2k)!},$$

RHS is a function on $\mathbb{R}$, thus we conclude that RHS is a function that extends LHS to $[0,1]$, let's check that LHS is bounded near 0:

$$\lim_{x\to 0} \frac{1-\cos x^2}{x^4} = \lim_{x\to 0} \frac{1}{2}\frac{\sin x^2}{x^2} = \frac{1}{2}.$$

Hence by the theorem we have just quoted,

$$\int_0^1 \frac{1-\cos x^2}{x^4}\,dx = \int_0^1 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^{4k-4}}{(2k)!}\,dx.$$

Now RHS is a proper integral, the rest is routine: check that the convergence is uniform on $[0,1]$ and switch the order of integral sign and summation sign. This is very straightforward by M-test.

Exercise 3 (2007 Midterm). Determine $\lim_{x\to 0^+} \frac{e^x-1}{\sqrt{x}}$. Prove carefully that

$$\int_0^1 \frac{e^x-1}{\sqrt{x}}\,dx = \sum_{k=1}^{\infty} \frac{2}{k!(2k+1)}.$$

Note that LHS is an improper integral.

Example 4.

(a) Show that $\sum_{k=1}^{\infty}(-1)^k \frac{e^{-kx}}{k}$ converges uniformly on $[0,\infty)$.

(b) (i) Show that the series $\sum_{k=1}^{\infty}(-1)^k \frac{(2-x)^k}{2^k k}$ converges uniformly on $[0,2]$.

(ii) Show that the series $\sum_{k=1}^{\infty}(-1)^k \frac{(2-x)^{k+\sqrt{5}}}{2^k k}$ converges uniformly on $[0,2]$.

Solution. (a) Write the series as

$$\sum_{k=1}^{\infty}(-1)^k \frac{(e^{-x})^k}{k}.$$

Since $x \in [0,\infty)$, we have $e^{-x} \in (0,1]$, by substitution property it is enough to show the power series

$$\sum_{k=1}^{\infty}(-1)^k \frac{x^k}{k}$$

converges uniformly on $[0,1]$. For this, we prove that the power series converges pointwise on $[0,1]$, after that Abel's Limit Theorem will guarantee the uniformity.

The radius of convergence is

$$\frac{1}{\overline{\lim}\sqrt[k]{|(-1)^k \frac{1}{k}|}} = 1.$$

Therefore the power series converges on $(-1,1)$, in particular, on $[0,1)$. When $x=1$, the power series becomes

$$\sum_{k=1}^{\infty}(-1)^k \frac{1}{k},$$

which also converges by Alternating Series Test, thus we are done.

(b) (i) We try to use Abel's Limit Theorem. Firstly we write the series as the standard form of power series as follows

$$\sum_{k=1}^{\infty}\frac{(x-2)^k}{2^k k},$$

its radius of convergence is 2, hence it converges on $(0,4)$, thus on $(0,2]$. At $x=0$, the power series becomes

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k},$$

which also converges by Alternating Series Test, so by Abel's Limit Theorem, done.

(ii) This time we cannot use Abel's Limit Theorem directly because the series is not a power series, as $k+\sqrt{5} \notin \mathbb{N}$ for every k. We observe that

$$\sum_{k=1}^{\infty}(-1)^k \frac{(2-x)^{k+\sqrt{5}}}{2^k k} = \underbrace{(2-x)^{\sqrt{5}}}_{\text{bounded on } [0,2]} \underbrace{\sum_{k=1}^{\infty}(-1)^k \frac{(2-x)^k}{2^k k}}_{\text{converges uniformly by part (a)}},$$

by Bounded Multiplier Property, done.

Exercise 4. It is given $1/(1+x^2) = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ for $x \in (-1,1)$, prove that

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

Exercise 5. Show that

$$1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \cdots = \frac{1}{4\sqrt{2}}(\pi + 2\ln(\sqrt{2}+1)).$$

Exercise 6. Dini's Theorem is difficult to use since the pointwise limit of a sequence of functions is rarely computable explicitly. One possible application can be seen in Practice Exercise 72 in the lecture notes of this course.

Show that in Dini's Theorem, the condition (i) that f_n is pointwise increasing **cannot be removed**.