Continuous Functions

We need to know

- how to apply the rigorous definition of continuity to solve problems;
- how to relate the concepts of continuity to limit of sequences;
- how to judge the existence of continuous functions satisfying certain properties, with the help of properties of continuous functions we know: Theorem 3 ~ 8.

Key definitions and results

Definition 1 (Limit of Functions). Let $f(x)$ be a function on S. For $x_0 \in \mathbb{R}$, we say that $\lim_{x\to x_0} f(x) = L$ if we have

$$\forall \epsilon > 0, \exists \delta > 0 \quad \text{s.t.} \quad 0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon.$$

Definition 2 (Continuity). A function $f(x)$ on S is said to be **continuous at $x_0 \in S$** if we have

$$\forall \epsilon > 0, \exists \delta > 0 \quad \text{s.t.} \quad |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon.$$

Remark. In Definition 2 we require $x_0 \in S$ while in Definition 1, x_0 needs not lie in S! This is the only difference of the two definitions. Moreover, we implicitly require $\boldsymbol{x \in S}$ in both definitions, we may write $\lim_{S\ni x\to x_0}$ for emphasis when taking limit.

Remark. The two sided limits are similarly defined with an obvious modification of Definition 1, we omit them for saving space.

Theorem 3 (Sequential Continuity). $f : S \to \mathbb{R}$ is continuous at $x_0 \in S$ if and only if for every $\{x_n\}$ s.t. $x_n \to x_0$ in S, $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Theorem 4 (Sign Preserving Property). If $g : S \to \mathbb{R}$ is continuous, $g(x_0) > 0$, then there is an interval $I = (x_0 - \delta, x_0 + \delta)$ with $\delta > 0$ such that

$$g > 0 \quad \text{on } S \cap I.$$

The case $g(x_0) < 0$ is similar by considering $-g$.

Theorem 5 (Intermediate Value). If $f : [a,b] \to \mathbb{R}$ is continuous and y_0 is between $f(a)$ and $f(b)$, then there is an $x_0 \in [a,b]$ such that $f(x_0) = y_0$.

Theorem 6 (Extreme Value). If $f : [a,b] \to \mathbb{R}$ is continuous, then there are $x_1, x_2 \in [a,b]$, such that

$$f(x_1) = \max_{x\in[a,b]} f(x) \quad \text{and} \quad f(x_2) = \min_{x\in[a,b]} f(x).$$

Theorem 7 (Continuous Injection). If $f(x)$ is continuous and injective on $[a,b]$, then $f(x)$ is strictly monotone on $[a,b]$ and

$$f([a,b]) = [f(a), f(b)] \text{ or } [f(b), f(a)].$$

Theorem 8 (Continuous Inverse). If f is continuous and injective on $[a,b]$, then f^{-1} is continuous on $f([a,b])$.

Example 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Show that f is **nowhere** continuous.

Sol Recall that a function $f : S \to \mathbb{R}$ is continuous at $x_0 \in S$ if and only if for every sequence $\{x_n\} \to x_0$ in S, $f(x_n) \to f(x_0)$.

To show $f(x)$ is discontinuous at x_0, it is enough to show there are two sequences converging to x_0, but their images converge to different limits.

Specifically, by density of $\mathbb{Q}$ there are $r_n \in \mathbb{Q}$, $r_n \to x_0$.

Similarly, by density of $\mathbb{R} \setminus \mathbb{Q}$, there are $w_n \in \mathbb{R} \setminus \mathbb{Q}$, $w_n \to x_0$.

Since $f(r_n) \to 1$ and $f(w_n) \to 0$, their images have different limits, so $f(x)$ is discontinuous at x_0. ∎

Example 2 (ϵ-δ Definition). Prove each of the following **using the ϵ-δ definition of continuity**:

(a) Show that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sqrt{|x+3|}$ is continuous on $\mathbb{R}$.

(b) Show that $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^3$ is continuous on $\mathbb{R}$

Sol **(a)** Let $x_0 \in \mathbb{R}$, we show that $f(x)$ is continuous at x_0.

For this, we analyse the expression $|f(x) - f(x_0)|$, which satisfies

$$|f(x) - f(x_0)| = |\sqrt{|x+3|} - \sqrt{|x_0+3|}| \le \sqrt{\big||x+3| - |x_0+3|\big|} \le \sqrt{|x - x_0|},$$

therefore by choosing $\delta = \epsilon^2$, we have

$$|x - x_0| < \delta = \epsilon^2 \implies |f(x) - f(x_0)| \le \sqrt{\epsilon^2} = \epsilon.$$

Remark. For people who have slightly advanced knowledge, one may guess $f(x)$ is **uniformly continuous**, this is indeed true! We shall come back to this later.

(b) Let $x_0 \in \mathbb{R}$, we analyse $|g(x) - g(x_0)| = |x - x_0||x^2 + xx_0 + x_0^2|$.

We expect when x is close to x_0, $x - x_0 \to 0$, but $|x^2 + xx_0 + x_0^2|$ can be large, and $\boxed{\text{small}} \times \boxed{\text{big}}$ can be big! We need to be careful in using the ϵ-δ definition.

Indeed the term $|x^2 + xx_0 + x_0^2|$ causes no trouble even it is large!

The choice of x_0 is fixed, it cannot be arbitrarily large.

As x is supposed to be close to x_0, let's say when $|x - x_0| < 1$, we have $|x| < 1 + |x_0|$, $|x|$ cannot be arbitrarily large when $|x - x_0| < 1$, so the whole term

$$|x^2 + xx_0 + x_0^2|$$

cannot be arbitrarily large as by triangle inequality thrice it has a bound

$$(1 + |x_0|)^2 + (1 + |x_0|)|x_0| + |x_0|^2 =: C(x_0)$$

when $|x - x_0| < 1$, where $C(x_0)$ is a constant depending on x_0, which is really a constant since x_0 is fixed.

Now for every $\delta < 1$, we have

$$|x - x_0| < \delta \implies |g(x) - g(x_0)| < \delta C(x_0),$$

Therefore for every $\epsilon > 0$, we may choose

$$\delta = \min\{1, \epsilon / C(x_0)\}$$

such that

$$|x - x_0| < \delta \implies |g(x) - g(x_0)| < \epsilon. \quad \blacksquare$$

Example 3.

(a) Can there be a continuous function $f(x)$ from $(0,1)$ **onto** $[0,1]$?

(b) Can there be a continuous function $g(x)$ from $[0,1]$ **onto** $(0,1)$?

Sol **(a)** Yes, $f(x) = \frac{1}{2}\sin(\pi x) + \frac{1}{2}$ is such a function.

(b) No, there can't be such a function.

Suppose a function $g : [0,1] \to (0,1)$ is onto and continuous, then for every $n \geq 2$, $\frac{1}{n} \in (0,1)$ and there is an $x_n \in [0,1]$ (by surjectivity) such that

$$f(x_n) = \frac{1}{n}.$$

Although $\frac{1}{n}$ converges, we cannot judge the convergence of $\{x_n\}$.

Fortunately, by Bolzano-Weierstrass (B-W) Theorem there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to a$ for some $a \in [0,1]$. Therefore for every k,

$$f(x_{n_k}) = \frac{1}{n_k},$$

if we take $k \to \infty$, then by Sequential Continuity Theorem,

$$f(a) = 0,$$

a contradiction. This solution is somewhat a copy of Extreme Value Theorem. ∎

Alternatively, if f is continuous and onto, then $f([0,1]) = (0,1)$ has a maximum and minimum by Extreme Value Theorem, this is impossible. ∎

Remark. By homework 1 there is a bijection from $[0,1]$ to $(0,1]$, but there **can't be <u>continuous</u>** bijection map from $[0,1]$ to $(0,1]$ since $(0,1]$ has no minimum.

Example 4 (Presentation Exercise 62). If $f(x) = x^3$, then $f(f(x)) = x^9$. Is there a contniuous function $g : [-1,1] \to [-1,1]$ such that

$$g(g(x)) = -x^9$$

for every $x \in [-1,1]$?

Sol Let's suppose such $g : [-1,1] \to [-1,1]$ exists and derive a contradiction.

First, let's observe g must be injective. Suppose that $g(x) = g(y)$, then $g(g(x)) = g(g(y))$, therefore $-x^9 = -y^9$, and thus $x = y$.

By Continuous Injection Theorem, g must be strictly monotone, but this leads us to a contradiction by considering the following cases.

Case 1. Suppose that $g(x)$ is strictly increasing, then

$$x > y \implies g(x) > g(y) \implies g \circ g(x) > g \circ g(y),$$

therefore $g \circ g(x) = -x^9$ is strictly increasing, a contradiction.

Case 2. Similarly, if $g(x)$ is strictly decreasing, then

$$x > y \implies g(x) < g(y) \implies g \circ g(x) > g \circ g(y),$$

so $g \circ g(x)$ is strictly increasing, again a contradiction.

Combining two cases above, we conclude that such g cannot exist. ∎

Example 5 (Mean-Value Theorem for Integrals). Let $f(x)$ be continuous on $[a,b]$ and $g(x)\ge 0$ be integrable on $[a,b]$. Show that there is $c\in[a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

<u>Sol</u> Let $m=\min_{x\in[a,b]} f(x)$ and $M=\max_{x\in[a,b]} f(x)$, such min and max exist by Extreme Value Theorem.

Since $g(x)\ge 0$, direct comparison gives

$$m\int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le M\int_a^b g(x)\,dx. \qquad (*)$$

Case 1. Suppose that $\int_a^b g(x)\,dx>0$, then

$$m\le \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}\le M,$$

and therefore by Intermediate Value Theorem, there is $c\in[a,b]$ such that

$$\frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}=f(c).$$

Case 2. Suppose that $\int_a^b g(x)\,dx=0$, then the method in case 1 fails since we cannot divide a number by zero, but by $(*)$ we have

$$\int_a^b f(x)g(x)\,dx=0,$$

therefore there is $c\in[a,b]$ such that $\int_a^b f(x)g(x)\,dx=0=f(c)\int_a^b g(x)\,dx$. Any choice $c\in[a,b]$ will do. ∎

Exercises

1. Let $f:[0,1]\to[0,1]$ be continuous, show that there must be an $x_0\in[0,1]$ such that $f(x_0)=x_0$.

2. Let $f:\mathbb{R}\to\mathbb{R}$ be continuous function such that $f(r+1/n)=f(r)$ for all $r\in\mathbb{Q}$ and $n\in\mathbb{N}$. Prove that f is a constant function.

3. Let $f,g:[a,b]\to\mathbb{R}$ be two continuous functions with

$$\sup_{x\in[a,b]} f(x)=M=\sup_{x\in[a,b]} g(x).$$

Show that there is $x_0\in[a,b]$ such that $f(x_0)=g(x_0)$.

4. Prove there is no differentiable function $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(f(x))=x^2-3x+3.$$

Hint: Consider fixed points and apply chain rule for differentiation.

5. Let $f:[a,b]\to\mathbb{R}$ be **Hölder's continuous** of order $\alpha>0$, i.e., there is a constant L such that

$$|f(x)-f(y)|\le L|x-y|^\alpha \quad \text{for all } x,y\in[a,b].$$

Show that if $\alpha>1$, then $f(x)$ must be a constant function.

Remark. Therefore the requirement $\alpha\in(0,1]$ is always imposed to rule out such triviality. When $\alpha=1$, $f(x)$ is said to be **Lipchitz continuous** and L is called a **Lipscthiz constant**.

6. Let $f:[a,b]\to\mathbb{R}$ be continuous. Suppose for every $x\in[a,b]$ there is a $y\in[a,b]$ such that $|f(y)|\le\frac{1}{2}|f(x)|$, prove that $f(c)=0$ for some $c\in[a,b]$.

7. Let $f(x)$ be continuous on $[0,1]$ with $f(0)=f(1)$. Show that for each $n\in\mathbb{N}$, there is a $\zeta\in[0,1-1/n]$ such that $f(\zeta+1/n)=f(\zeta)$.

8. Let $f(x)$ be continuous on $[a,b]$. Show that $M:[a,b]\to\mathbb{R}$ defined by $M(x):=\sup\{f(t):a\le t\le x\}$ is continuous on $[a,b]$.

9. **(2006 Spring)** Let $f(x),g(x):[0,1]\to\mathbb{R}$ be continuous. If there are $x_1,x_2,\dots\in[0,1]$ such that $g(x_n)=f(x_{n+1})$, prove that there is $w\in[0,1]$ such that $g(w)=f(w)$.

Caution: $\{x_{n_i}\}$ converges does not imply $\{x_{n_i+1}\}$ converges.

10. Let $f,g:[0,1]\to[0,1]$ be continuous functions such that $f(g(x))=g(f(x))$ for every $x\in[0,1]$. Prove that $f(x)=g(x)$ must have a solution.