Sequence and Series of Functions; Uniform and Nonuniform Convergence

Key Definitions and Results

Definition 1. A sequence of functions $f_n : E \to \mathbb{R}$ is said to **converges pointwise on E to a function $f : E \to \mathbb{R}$** if for every $x \in E$,

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Definition 2. A sequence of functions $f_n : E \to \mathbb{R}$ is said to **converges uniformly on E to a function $f : E \to \mathbb{R}$** if

$$\lim_{n\to\infty} \|f_n - f\|_E := \lim_{n\to\infty}\left(\sup\{|f_n(x) - f(x)| : x \in E\}\right) = 0.$$

In this case we denote $f_n \rightrightarrows f$ **on E**, for short.

Equivalently, we say that $f_n \rightrightarrows f$ **on E** if for every $\epsilon > 0$, there is an N such that

$$n > N \implies |f_n(x) - f(x)| < \epsilon \quad \text{for all } x \in E.$$

Theorem 3. Consider the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, the number

$$R := \frac{1}{\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|}}$$

is called the **radius of convergence of $\sum a_n(x-c)^n$**. Define $\frac{1}{0} = \infty, \frac{1}{\infty} = 0$.

(i) If $R = 0$, the series converges at $x = c$ and diverges elsewhere.

(ii) If $0 < R < \infty$, the series converges **absolutely** when $|x - c| < R$ and diverges when $|x - c| > R$.

(iii) If $R = \infty$, the series converges **absolutely** on $\mathbb{R}$.

Theorem 4 (M-test). Let $g_n : E \to \mathbb{R}$ be a sequence of functions on E. Suppose:

(i) For each n there is an $M_n \in \mathbb{R}$ such that $|g_n(x)| \le M_n$ on E.

(ii) $\sum_{n=1}^{\infty} M_n$ converges.

Then $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly on E.

Theorem 5 (Uniform Cauchy Criterion). Let $f_1, f_2, \cdots : E \to \mathbb{R}$ be a sequence of functions, then the following are equivalent:

(i) $\{f_n\}$ converges uniformly on E.

(ii) $\{f_n\}$ is **uniformly Cauchy**: For every $\epsilon > 0$, there is an N such that

$$m, n > N \implies \|f_m - f_n\|_E < \epsilon.$$

Example 1. Let $a_0, a_1, a_2, \cdots \in \mathbb{R}$.

(a) If $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges at $x_0 \neq 0$, then it converges absolutely when $|x| < |x_0|$.

(b) If $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ diverges at $x_0 \neq 0$, then it diverges when $|x| > |x_0|$.

Solution. (a) If $\sum a_n x_0^n$ converges, then $a_n x_0^n$ is bounded. Let $|a_n x_0^n| \le M$ for all n, then for every x s.t. $|x| < |x_0|$ we have

$$|a_n x^n| = \left| a_n x_0^n \left(\frac{x}{x_0}\right)^n \right| \le M \left|\frac{x}{x_0}\right|^n,$$

therefore since $\sum |\frac{x}{x_0}|^n$ converges, $\sum a_n x^n$ converges by comparison test and absolute convergence test.

(b) Let $|x| > |x_0|$, if $\sum a_n x^n$ converges, so does $\sum a_n x_0^n$ by part (a), this is a contradiction.

Remark. In part (a) the series $\sum a_n x^n$ converges uniformly on any $[-r, r]$, $r < |x_0|$, by M-test. In fact for any $0 < r < |x_0|$ it converges uniformly on $[-r, |x_0|]$ by Abel's Limit Theorem in the next tutorial.

Example 2. Find the radius of convergence R of the power series

$$\sum_{n=0}^{\infty} \frac{n!}{n^n} x^n.$$

Can the power series converge at $x = \pm R$?

Solution. To find the radius of convergence we need to compute

$$\overline{\lim} \sqrt[n]{\frac{n!}{n^n}}.$$

Actually the limit exists, so $\overline{\lim} = \lim$. To see this, recall that

$$\underline{\lim} \left| \frac{a_{n+1}}{a_n} \right| \leq \underline{\lim} \sqrt[n]{|a_n|} \leq \overline{\lim} \sqrt[n]{|a_n|} \leq \overline{\lim} \left| \frac{a_{n+1}}{a_n} \right|,$$

hence if the limit of the nth root of a_n is difficult to compute, let's first try to compute the limit of its ratio. **If** $\lim \boldsymbol{a_{n+1}/a_n}$ **exists**, then by the above inequality, **so does** $\lim \sqrt[n]{a_n}$, moreover, $\lim \sqrt[n]{a_n} = \lim \boldsymbol{a_{n+1}/a_n}$.

Bearing this in mind, we have

$$\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \Big/ \frac{n!}{n^n} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} \implies \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \frac{1}{e},$$

and thus the radius of convergence is $R = e$.

At the point $x = \pm R = \pm e$, note that $|n!(\pm e)^n/n^n| = n!e^n/n^n$. By Stirling formula,

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \implies \frac{n!e^n}{n^n} \sim \sqrt{2\pi n} \to \infty,$$

therefore the power series diverges at $x = \pm e$ by term test.

Exercise 1. Find the radius of convergence for the following power series:

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n} x^n$; (b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{2^{n^2}}$; (c) $\displaystyle\sum_{n=1}^{\infty} n^n x^n$; (d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + (-1)^n}{n^2} (x-3)^n$;

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{a^n}{n+1} x^{2n}$; (f) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{n+1}{n}\right)^{n^2} x^{n^2}$.

Exercise 2. Let $\{a_n\}$ be a sequence of positive numbers such that $\sum_{n=0}^{\infty} a_n$ diverges. Suppose also that

$$\lim_{n\to\infty} \frac{a_n}{a_0 + a_1 + a_2 + \cdots + a_n} = 0,$$

show that the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 1.

Example 3. Study the uniform convergence of

$$f_n(x) = \frac{nx}{1+n^2x^2}$$

for $n \geq 1$, on the interval $(0,1)$ and $(1,\infty)$ respectively.

Solution. Easy to see that $f_n \to 0$ pointwise on $(0,\infty)$.

On $(0,1)$, since

$$\|f_n - 0\|_{(0,1)} \geq \left| f_n\left(\frac{1}{n}\right) \right| = \frac{1}{1+1} = \frac{1}{2},$$

thus $\|f_n\|_{(0,1)} \not\rightrightarrows 0$.

On $(1,\infty)$, since for every $x \in (1,\infty)$,

$$|f_n(x) - 0| \leq \frac{nx}{n^2x^2} = \frac{1}{nx} \leq \frac{1}{n},$$

thus $\|f_n - 0\|_{(1,\infty)} \leq \frac{1}{n}$, hence $f_n \rightrightarrows 0$ on $(1,\infty)$.

Exercise 3 (HKU, Analysis I Final, Fall 2004). Consider the sequence of functions

$$f_n(x) = \frac{1}{n} e^{-n^2x^2}, \quad x \in \mathbb{R}, n \geq 1.$$

(a) Show that f_n converges uniformly to some differentiable function f on $\mathbb{R}$.

(b) Show that f_n' converges to f' pointwise on $\mathbb{R}$.

(c) Determine if the convergence f_n' is uniform on any bounded interval $[a,b] \subset \mathbb{R}$. Justify your assertion.

Exercise 4. Let $k \geq 0$ be an integer and define a sequence of functions as

$$f_n(x) = \frac{x^k}{x^2+n}, \quad x \in \mathbb{R}, n \geq 1.$$

For which values of k does the sequence converge uniformly on $\mathbb{R}$? On every bounded subset of $\mathbb{R}$?

Exercise 5 (2010 Midterm, Q3(b)). Define $f_1, f_2, \cdots : (0,1) \to \mathbb{R}$ by

$$f_n(x) = \frac{e^x}{x\sqrt{n} + \sin^2(nx - 25)}.$$

Prove that f_n converges pointwise on $(0,1)$. Determine with proof if f_n converges uniformly on $(0,1)$.

Example 4. Show that

$$\sum_{n=1}^{\infty} x^n (1-x)^2$$

converges uniformly on $[0,1]$.

Solution. Let $g_n(x) = x^n(1-x)^2$. There are two ways to find a sharp enough upper bound of $|g_n(x)|$.

Method 1. We consider the derivative

$$g_n'(x) = (1-x)x^{n-1}(n-(n+2)x).$$

If $g_n'(x) = 0$, then $x = 0, 1$ or $n/(n+2)$ (< 1). Since $g_n(0) = g_n(1) = 0$, and since $g_n(n/(n+2)) > 0$ is the only local extreme value of g on $(0,1)$, it must be a global maximum. So for any $x \in [0,1]$,

$$|g_n(x)| = g_n(x) \le g_n\left(\frac{n}{n+2}\right) = 4\left(\frac{n}{n+2}\right)^n \frac{1}{(n+2)^2} \le \frac{4}{(n+2)^2}.$$

Method 2. We may use AM-GM inequality. For $x \in [0,1]$,

$$\begin{aligned}
|g_n(x)| &= x^n(1-x)^2 \\
&= 4n^n \left(\frac{x}{n}\right)^n \left(\frac{1}{2} - \frac{x}{2}\right)^2 \\
&\le 4n^n \left(\frac{\overbrace{\frac{x}{n} + \cdots + \frac{x}{n}}^{n \text{ terms}} + (\frac{1}{2} - \frac{x}{2}) + (\frac{1}{2} - \frac{x}{2})}{n+2}\right)^{n+2} \\
&= \frac{4n^n}{(n+2)^{n+2}} \\
&\le \frac{4}{(n+2)^2}.
\end{aligned}$$

Therefore, as $\sum_{n=1}^{\infty} 4/(n+2)^2$ converges, by M-test $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly on $[0,1]$.

Example 5.

(a) Show that $\sum_{k=0}^{\infty} x^k \not\rightrightarrows \frac{1}{1-x}$ on $(-1,1)$.

(b) Define $f_n(x) = x^n$ for $x \in [0,1]$ and $n \ge 1$. Show that $f_n \not\rightrightarrows 0$ on $[0,1)$.

Solution. (a) On $(-1,1)$ we have

$$\left|\sum_{k=0}^{n} x^k - \frac{1}{1-x}\right| = \left|\sum_{k=n+1}^{\infty} x^k\right| = \left|\frac{x^{n+1}}{1-x}\right|.$$

Can this quantity be uniformly small for sufficiently large n? No! Observe that for each integer n,

$$\left|\frac{x^{n+1}}{1-x}\right| \to \infty \quad \text{as } x \to 1^-.$$

In particular, for each n, there is an $x_n \in (-1,1)$, close enough to 1, such that $\left|\frac{x_n^{n+1}}{1-x_n}\right| \ge 1$, thus for each n,

$$\left\|\sum_{k=0}^{n} x^k - \frac{1}{1-x}\right\|_{(-1,1)} \ge \left|\frac{x_n^{n+1}}{1-x_n}\right| \ge 1,$$

which cannot converges to 0.

(b) The pointwise limit of f_n on $[0,1)$ is 0. But the quantity

$$|f_n(x)| = x^n$$

cannot be uniformly small for sufficiently large n because $\lim_{x\to 1^-} x^n = 1$. Specifically, for each n there is $x_n \in [0,1)$ such that $x_n^n > 1/2$, so

$$\|x^n - 0\|_{[0,1)} \ge x_n^n > 1/2 \implies x^n \not\rightrightarrows 0.$$

Example 6 (Lecture Notes p. 85, Slightly Modified). Show that the series

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$$

does not converge uniformly on $(0,1)$ by showing the uniform Cauchy criterion fails.

Solution. Let $f_n = \sum_{k=0}^{n} x^2/(1+x^2)^k$, we consider the difference

$$\begin{aligned} |f_{2n}(x) - f_n(x)| &= \sum_{k=n+1}^{2n} \frac{x^2}{(1+x^2)^k} \\ &\geq \sum_{k=n+1}^{2n} \frac{x^2}{(1+x^2)^{2n}} \\ &= \frac{nx^2}{(1+x^2)^{2n}}. \end{aligned}$$

For each $n > 1$ we let $x = x_n = 1/\sqrt{n} \in (0,1)$ such that

$$\|f_{2n} - f_n\|_{(0,1)} \geq |f_{2n}(x_n) - f_n(x_n)| = \frac{1}{(1+\frac{1}{n})^{2n}} \to \frac{1}{e^2},$$

which shows $\{f_n\}$ cannot be uniformly Cauchy on $(0,1)$.

Exercise 6. Study the uniform convergence of

$$\sum_{k=1}^{\infty} \frac{kx}{(1+x)(1+2x)\cdots(1+kx)}$$

on $[0,\lambda]$ and $[\lambda,\infty)$ respectively, where $\lambda > 0$.

Hint. On one interval the series converges uniformly. On another interval the convergence cannot be uniform, which can be shown by imitating the proof in the solution of the above example.