More Examples of Limit Superior and Limit Inferior

Key Definitions and Results

Theorem 1. Let $a_1, a_2, \cdots \in \mathbb{R}$, and $r \in \mathbb{R}$, then:

(i) If $\overline{\lim}\, a_n < r$, then $a_n < r$ for all but finitely many n.

(ii) If $r < \underline{\lim}\, a_n$, then $r < a_n$ for all but finitely many n.

Example 1. Let $a_1 \ge a_2 \ge a_3 \ge \cdots > 0$. Show that

$$\sum_{n=1}^{\infty} a_n \text{ converges} \implies \lim_{n\to\infty} n a_n = 0.$$

Solution. Since $\sum a_k$ converges, by Cauchy criterion for any $\epsilon > 0$ there is an N such that

$$n > m > N \implies \sum_{k=m+1}^{n} a_k < \epsilon.$$

Since $\{a_k\}$ is decreasing, we have

$$m,n > N \implies \frac{n-m}{n} \cdot n a_n = (n-m) a_n < \epsilon.$$

By taking $\overline{\lim}_{n\to\infty}$ on both sides, we have

$$\overline{\lim_{n\to\infty}}\, n a_n = \left(\lim_{n\to\infty} \frac{n-m}{n}\right) \cdot \left(\overline{\lim_{n\to\infty}}\, n a_n\right) = \overline{\lim_{n\to\infty}} \left(\frac{n-m}{n} \cdot n a_n\right) \le \epsilon.$$

Therefore we have $0 \le \overline{\lim}\, n a_n \le \epsilon$ for every $\epsilon > 0$. By taking $\epsilon \to 0^+$ we have

$$\overline{\lim}\, n a_n = 0,$$

this says that $\lim_{n\to\infty} n a_n = 0$ since $n a_n \ge 0$.

Exercise 1. Let $0 < a_1 < a_2 < \cdots$ be unbounded and set $s = \overline{\lim} \dfrac{\ln n}{\ln a_n}$. Let $t > 0$, show that the series $\displaystyle\sum_{n=1}^{\infty} a_n^{-t}$ converges for $t > s$ and diverges for $t < s$.

Example 2. Let $b_1, b_2, \cdots \ge 0$ be such that $\lim_{n\to\infty} b_n = 0$. Let $\lambda \in (0,1)$, define $a_1 \ge 0$ and for $n \ge 1$ define

$$a_{n+1} = b_n + \lambda a_n.$$

(a) Explain why the following argument **is wrong**:

> We take $\overline{\lim}$ on both sides to get
>
> $$\overline{\lim}\, a_n = \overline{\lim}(b_n + \lambda a_n) = \lim b_n + \overline{\lim}\, \lambda a_n = \lambda\, \overline{\lim}\, a_n.$$
>
> It follows that $(1-\lambda)\overline{\lim}\, a_n = 0$ and thus $\overline{\lim}\, a_n = 0$. Since $a_n \ge 0$, we conclude $\lim_{n\to\infty} a_n = 0$.

(b) Prove that indeed $\lim_{n\to\infty} a_n = 0$.

Solution. (a) We don't have $(1-\lambda)\overline{\lim}\, a_n = 0$ since $\overline{\lim}\, a_n$ is possibly ∞, in this case we cannot subtract $\overline{\lim}\, a_n$ on both sides of $\overline{\lim}\, a_n = \lambda\, \overline{\lim}\, a_n$.

(b) Since $b_n \ge 0$, by induction $a_n \ge 0$ for each n. For every $\epsilon > 0$, there is an N such that

$$k \ge N \implies a_{k+1} < \epsilon + \lambda a_k.$$

By dividing λ^{k+1} on both sides, we have

$$\frac{a_{k+1}}{\lambda^{k+1}} < \frac{\epsilon}{\lambda^{k+1}} + \frac{a_k}{\lambda^k},$$

by taking $\sum_{k=N}^{n-1}$ on both sides we have

$$\frac{a_n}{\lambda^n} - \frac{a_N}{\lambda^N} < \epsilon \sum_{k=N}^{n-1} \frac{1}{\lambda^{k+1}} = \frac{1}{\lambda^n} + \frac{1}{\lambda^{n-1}} + \cdots + \frac{1}{\lambda^{N+1}},$$

therefore

$$a_n < \lambda^{n-N} a_N + \epsilon(1 + \lambda + \cdots + \lambda^{n-N-1}) < \lambda^{n-N} a_N + \frac{\epsilon}{1-\lambda}.$$

By taking $\overline{\lim}$ on both sides we have

$$\overline{\lim}\, a_n \le \frac{\epsilon}{1-\lambda}.$$

Since $\epsilon > 0$ is arbitrary, $\overline{\lim}\, a_n = 0$, and thus $\lim_{n\to\infty} a_n = 0$.

Example 3. Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive numbers such that $\sum_{n=1}^{\infty} b_n = \infty$, show that

$$\overline{\lim_{n\to\infty}} \frac{a_1+a_2+\cdots+a_n}{b_1+b_2+\cdots+b_n} \le \overline{\lim_{n\to\infty}} \frac{a_n}{b_n}.$$

Solution. For any $\epsilon > 0$, we have

$$\overline{\lim} \frac{a_n}{b_n} < \underbrace{\overline{\lim} \frac{a_n}{b_n}}_{:=\alpha} + \epsilon = \alpha + \epsilon,$$

thus by Theorem 1 there is an N such that

$$k > N \implies \frac{a_k}{b_k} < \alpha + \epsilon.$$

Thus we have $a_k < (\alpha+\epsilon)b_k$ and

$$\begin{aligned}\frac{a_1+\cdots+a_n}{b_1+\cdots+b_n} &= \frac{a_1+\cdots+a_N+a_{N+1}+\cdots+a_n}{b_1+\cdots+b_n}\\ &< \frac{a_1+\cdots+a_N}{b_1+\cdots+b_n} + \frac{(\alpha+\epsilon)(b_{N+1}+\cdots+b_n)}{b_1+\cdots+b_n}.\end{aligned}$$

By taking $\overline{\lim}$ on the ends of the above inequality we have

$$\overline{\lim} \frac{a_1+\cdots+a_n}{b_1+\cdots+b_n} \le 0 + (\alpha+\epsilon)\cdot 1 = \alpha+\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we taking $\epsilon \to 0^+$ to get

$$\overline{\lim} \frac{a_1+\cdots+a_n}{b_1+\cdots+b_n} \le \alpha = \overline{\lim} \frac{a_n}{b_n}.$$

Exercise 2. Let $x_1, x_2, \cdots \in \mathbb{R}$, show that

$$\underline{\lim}_{n\to\infty} x_n \le \underline{\lim}_{n\to\infty} \ln\left(\frac{e^{x_1}+e^{x_2}+\cdots+e^{x_n}}{n}\right).$$

Exercise 3 (2003 Midterm (L1)). Let $\{a_n\}$ be a sequence of real numbers. show that

$$\limsup_{n\to\infty} \frac{a_n}{n^2} \le \limsup_{n\to\infty} \frac{a_{n+1}-a_n}{2n+1}.$$

Example 4 (2006 Midterm). Let $\{x_n\}$ be a bounded sequence of real numbers such that

$$\lim_{n\to\infty}(x_{n+1}-x_n) = 0, \quad \liminf_{n\to\infty} x_n = a \quad \text{and} \quad \limsup_{n\to\infty} x_n = b.$$

Show for every $c \in [a,b]$, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $\lim_{i\to\infty} x_{n_i} = c$.

Solution. Let $c \in [a,b]$. We further assume $c \ne a,b$ as otherwise we are done. Now we need to show that in every **small** neighborhood $(c-r, c+r)$ of $c^{(*)}$, we can always find an element in $\{x_1, x_2, \dots\}$.

To do this, let's for the sake of contradiction suppose there is a small neighborhood $(c-r,c+r)$ of c which contains **none** of x_n's. Then by the condition $\lim_{n\to\infty}(x_{n+1}-x_n) = 0$, there is an $N \in \mathbb{N}$,

$$n > N \implies |x_{n+1}-x_n| < r.$$

Now the sequence $\{x_n\}_{n>N}$ cannot jump too far away from x_n to x_{n+1}.

Case 1 ($x_{N+1} \in [a, c-r]$). In this case,

$$\begin{aligned} x_{N+1} \in [a,c-r] &\implies x_{N+2} \in [a,c-r]\\ &\implies x_{N+3} \in [a,c-r] \implies \cdots \implies x_n \in [a,c-r]\end{aligned}$$

for every $n \ge N+1$, so $\overline{\lim}\, x_n \le c-r < b$, a contradiction.

Case 2 ($x_{N+1} \in [c+r, b]$). In this case the argument in case 1 carries over, and we arrive to the contradiction that $\underline{\lim}\, x_n \ge c+r > a$.

In conclusion, every small neighborhood of c contains one of x_n's. Let $K \in \mathbb{N}$ be such that

$$i > K \implies \left(c - \frac{1}{i}, c + \frac{1}{i}\right) \subseteq [a,b],$$

then there is an $x_{n_i} \in (c-\frac{1}{i}, c+\frac{1}{i})$. Now $\lim_{i\to\infty} x_{n_i} = c$.

Exercise 4. Let $a_1, a_2, a_3, \cdots > 0$.

(a) Prove that $\overline{\lim_{n\to\infty}} \left(\frac{a_1+a_{n+1}}{a_n}\right)^n \ge e$. Recall that $e^{-1} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$.

(b) Show that the bound e in the above estimate is optimal.

Exercise 5. Prove the following generalization of Exercise 4: For a positive integer p and a positive sequence $\{a_n\}_{n=1}^{\infty}$, prove that $\overline{\lim_{n\to\infty}} \left(\frac{a_1+a_{n+p}}{a_n}\right)^n \ge e^p$.

Hint: The proof is more or less the same as the previous exercise.

(*) Let's define, by small, to mean $(c-r, c+r) \subseteq [a,b]$.