Sequences

We need to know

- the usual strategy to show typical sequences converge;
- how to apply the rigorous definition of limit of sequences.

Key definitions and results

Definition 1 (Convergence of Seqeucne). We say that $\{a_n\}$ **(or a_n) converges to a**, denoted by $\lim_{n\to\infty} a_n = a$ or $a_n \to a$, if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \quad \text{s.t.} \quad n > N \implies |a_n - a| < \epsilon.$$

Remark. In Definition 1 we don't treat $a = \infty$ as a limit of sequence.

Definition 2 (Divergence to $\pm\infty$).

- $\{a_n\}$ **diverges to $+\infty$**, denoted by $\lim_{n\to\infty} a_n = +\infty$, if
$$\forall b > 0, \exists N \in \mathbb{N} \quad \text{s.t.} \quad n > N \implies a_n > b.$$
- $\{a_n\}$ **diverges to $-\infty$**, denoted by $\lim_{n\to\infty} a_n = -\infty$, if
$$\forall b < 0, \exists N \in \mathbb{N} \quad \text{s.t.} \quad n > N \implies a_n < b.$$

Definition 3 (Bounded Sequence). A sequence $\{a_n\}$ is **bounded** if there is an $M > 0$ such that $|a_n| \le M$ for each n.

Definition 4 (Cauchy Sequence). A sequence $\{a_n\}$ is a **Cauchy sequence** if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \quad \text{s.t.} \quad m, n > N \implies |x_m - x_n| < \epsilon.$$

Theorem 5 (Sandwich). If for each n, $x_n \le z_n \le y_n$, then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = z \implies \lim_{n\to\infty} z_n = z.$$

Theorem 6 (Bolzano-Weierstrass). If $\{x_n\}$ is bounded, then it has a convergent subsequence, denoted by $\{x_{n_k}\}$.

Remark. We treat n_k as a **strictly** increasing function from $\mathbb{N} \to \mathbb{N}$ in k, say $n_k = 2k$.

Theorem 7 (Cauchy). $\{x_n\}$ converges if and only if it is a Cauchy sequence.

Example 1 (2009 Fall). Let $x_1 = 1$ and for $n = 1, 2, \ldots$ let

$$x_{n+1} = \frac{x_n^3 + x_n}{5}. \qquad (*)$$

Prove that the sequence $\{x_n\}$ converges and find its limit.

Sol
- We show that $\{x_n\}$ is decreasing by MI.

When $N = 1$,
$$x_2 = \frac{1+1}{5} < x_1 = 1.$$
Assume $x_n < x_{n-1}$, then
$$x_{n+1} = \frac{x_n^3 + x_n}{5} < \frac{x_{n-1}^3 + x_{n-1}}{5} = x_n,$$
therefore by MI, $\{x_n\}$ is decreasing.

- Since x_n is bounded below by 0, $x_n \to a$ for some $a \in \mathbb{R}$.
- By taking $\lim_{n\to\infty}$ on both sides of $(*)$,
$$a = \frac{a^3 + a}{5} \iff a(a+2)(a-2) = 0$$
iff $a = 0, -2$ or $a = 2$, and the choice 2 and -2 are rejected since $0 \le x_n \le x_1 = 1$ for every $n \ge 1$, therefore
$$\lim_{n\to\infty} x_n = 0. \qquad ∎$$

Remark. Sequences of the form $x_{n+1} = P(x_n)$ with $P(x)$ being **obviously increasing** can be likely tackled in the same way.

Example 2 (2009 Fall). Let $x_1 = 2$ and for $n = 1, 2, \ldots$, let

$$x_{n+1} = \frac{22}{3} + \frac{16}{3x_n}.$$

(a) Prove that the sequence $\{x_n\}$ converges and find its limit.

(b) Prove that the series $\sum_{n=1}^{\infty}(x_n - x_{n+1})$ converges and determine its sum.

Sol **(a)** The numerical experiment shows us

$$0 < x_{2n-1} < x_{2n+1} < x_{2n+2} < x_{2n}.$$

Having formulated this statement, we can prove this by induction on n. This is standard and has been done in Tutorial Note 0.

The *even sequence* $\{x_{2n}\}$ is shown to be decreasing. The induction shows that $x_{2n} > 0$ for every n, hence it is bounded below, and thus convergent.

The *odd sequence* $\{x_{2n-1}\}$ is increasing. By

$$x_{2n-1} \le x_{2n+2} < x_{2n} < x_{2n-2} < \cdots < x_2$$

for every n, so $\{x_{2n-1}\}$ is bounded above, and hence convergent.

Thus it makes sense to define

$$a = \lim_{n\to\infty} x_{2n-1} \quad \text{and} \quad b = \lim_{n\to\infty} x_{2n}.$$

To show $\{x_n\}$ converges, we need to show $a = b$. By taking $n = 2k$ and $n = 2k-1$ respectively in the definition of recursive relation, we have

$$x_{2k+1} = \frac{22}{3} + \frac{16}{3x_{2k}} \quad \text{and} \quad x_{2k} = \frac{22}{3} + \frac{16}{3x_{2k-1}}.$$

By taking $k \to \infty$, we have

$$a = \frac{22}{3} + \frac{16}{3b} \quad \text{and} \quad b = \frac{22}{3} + \frac{16}{3a}.$$

The former equation gives $3ab = 22b + 16$. The latter one gives $3ab = 22a + 16$, equating them we have $a = b$, therefore $\{x_n\}$ converges.

As $a = b$, we have $a = \frac{22}{3} + \frac{16}{3a}$, we solve it to get $a = -2/3$ (rej.) or $a = 8$, we conclude $\lim_{n\to\infty} x_n = a = 8$.

(b) The series is called a telescoping series since its partial sum can be computed explicitly as follows: $\sum_{n=1}^{N}(x_n - x_{n+1}) = x_1 - x_{N+1} = 2 - x_{N+1}$. By taking $N \to \infty$, we have

$$\sum_{n=1}^{\infty}(x_n - x_{n+1}) = 2 - 8 = -6. \qquad \blacksquare$$

Example 3. Show that the sequence

$$a_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$$

converges by showing it is a Cauchy sequence.

Remark. The limit $\gamma := \lim_{n\to\infty} a_n = 0.5772156649$ is called the **Euler constant**.

Sol By definition we have

$$\begin{aligned} &a_{k+1} - a_k \\ &= \left(1 + \frac{1}{2} + \cdots + \frac{1}{k+1} - \ln(k+1)\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{k} - \ln k\right) \\ &= \frac{1}{k+1} + \ln\frac{k}{k+1} \\ &= \frac{1}{k+1} + \ln\left(1 - \frac{1}{k+1}\right). \end{aligned}$$

To proceed we try to approximate $\ln(1+x)$ for small x. By Taylor expansion we have for small x,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = x - \frac{x^2}{2}\Big(1 + \underbrace{(-\frac{2}{3}x + \cdots)}_{\text{call this } g(x)}\Big) = x - \frac{x^2}{2}(1 + g(x)),$$

where $g(x) \to 0$ as $x \to 0$. The expression on the RHS is obtained by factoring out the term $\frac{x^2}{2}$ for each of higher order terms.

If we replace x by $-\frac{1}{k+1}$, we have

$$a_{k+1} - a_k = \frac{1}{k+1} + \ln\left(1 - \frac{1}{k+1}\right) = -\frac{1}{2}\frac{1}{(k+1)^2}\left(1 + g\left(-\frac{1}{k+1}\right)\right).$$

Since $1 + g(-\frac{1}{k+1}) \to 1$ as $k \to \infty$, the sequence $\{1 + g(-\frac{1}{k+1})\}$ is bounded, thus there is $C > 0$,

$$|a_{k+1} - a_k| = \left|\frac{1}{k+1} + \ln\left(1 - \frac{1}{k+1}\right)\right| \le C\frac{1}{(k+1)^2}$$

for every $k \ge 1$.

Now for every $m > n$, we have

$$|a_m - a_n| = \left|\sum_{k=1}^{m}(a_{k+1} - a_k)\right| \le \sum_{k=n}^{m-1} |a_{k+1} - a_k| \le C\sum_{k=n}^{m-1} \frac{1}{(k+1)^2}.$$

The first inequality comes from triangle inequality, and the first equality comes from telescoping series property.

Since the series $\sum \frac{1}{(k+1)^2}$ converges, $\{\sum_{k=1}^{n} \frac{1}{(k+1)^2}\}$ is a Cauchy sequence.

Therefore for every $\epsilon > 0$, there is an N such that

$$m > n > N \implies \sum_{k=n}^{m-1} \frac{1}{(k+1)^2} < \frac{\epsilon}{C},$$

thus

$$m > n > N \implies |a_m - a_n| \le C \sum_{k=n}^{m} \frac{1}{(k+1)^2} < \epsilon.$$

Therefore $\{a_n\}$ is a Cauchy sequence. ∎

Alternative Method Without Cauchy Criterion. We also have a high-school argument in showing the convergence of $a_k := \sum_{i=1}^{k} \frac{1}{k} - \ln k$. By direct comparison we have

$$\sum_{k=2}^{n} \frac{1}{k} \le \underbrace{\sum_{k=2}^{n} \int_{k-1}^{k} \frac{1}{x}\, dx}_{=\ln n} \le \sum_{k=2}^{n} \frac{1}{k-1},$$

by separating the left and right inequalities, we have

$$\frac{1}{n} \le \sum_{k=1}^{n} \frac{1}{k} - \ln n \le 1,$$

therefore $a_n := \sum_{k=1}^{n} \frac{1}{k} - \ln n$ is both bounded above and below.

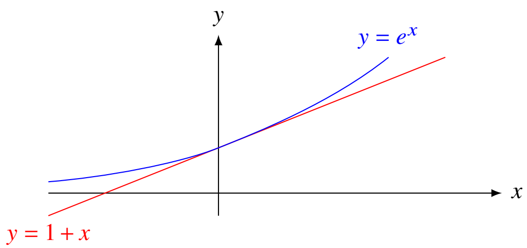
Next,

$$a_{n+1} - a_n = \frac{1}{n+1} + \ln\left(1 - \frac{1}{n+1}\right),$$

and actually RHS is ≤ 0. To see this, recall the convexity-like inequality: For every $x > -1$,

$$1 + x \le e^x \implies \ln(1+x) \le x.$$

This inequality is easily observed since $f(x) = e^x$ is convex and $1 + x$ is the tangent to $y = e^x$ at $x = 0$:



Therefore $a_{n+1} - a_n = \frac{1}{n+1} + \ln(1 - \frac{1}{n+1}) \le \frac{1}{n+1} + (-\frac{1}{n+1}) = 0.$ ∎

Exercises

We just list a few sequences of standard form here, more exercises of this type can be found in Dr Li's presentation problems assigned for students.

1. **(2008 Fall)** Let $x_1 = 1$ and for $n \ge 1$ define

$$x_{n+1} = \frac{4\sqrt{x_n} + x_n}{3}.$$

Show that $\{x_n\}$ converges and find its limit.

2. **(Rudin, p.81)** Let $a > 0$ and $x_0 > 0$, Let $x_n \ge 1$ be defined by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Show that $\{x_n\}$ converges to $\sqrt{a}$.

3. **(Rudin, p.81)** Fix $\alpha > 1$, take $x_1 > \sqrt{\alpha}$ and for $n \ge 1$ define

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n}.$$

Show that $\{x_n\}$ converge and find its limit.

Hint: Note that $1 + x_{n+1} = 2 + \frac{\alpha - 1}{1 + x_n}$, if we let $u_n = 1 + x_n$, then $u_n = 2 + \frac{\alpha - 1}{u_n}$, which is very very similar to Example 2. You may now imitate the solution there.

4. Let $a_1 = 1$. If $a_{n+1} = a_n + \dfrac{1}{a_n}$ for $n \ge 1$, prove that $\lim\limits_{n\to\infty} a_n = \infty$.

5. Let $\{x_n\}$ converge and define $y_n = n(x_n - x_{n-1})$ for $n \ge 2$. Is $\{y_n\}$ necessarily convergent? If $\{y_n\}$ converges, show that $y_n \to 0$.

6. Let $a_1 \ge a_2 \ge \cdots > 0, s_n = a_1 + a_2 + \cdots + a_n$ and $b_n = \dfrac{1}{a_{n+1}} - \dfrac{1}{a_n}$. Prove that if the sequence $\{s_n\}$ converges, then the sequence $\{b_n\}$ is unbounded.

In the following the rigorous definition of convergence may/must be involved.

7. Suppose $x_1, x_2, \dots \ge 0$ and $\lim\limits_{n\to\infty} (-1)^n x_n$ exists, show that $\lim\limits_{n\to\infty} x_n$ also exists.

8. Show that if both $\{a_n\}$ and $\{b_n\}$ are bounded, then there is a sequence of integers $n_1 < n_2 < n_3 < \cdots$ such that $\{a_{n_k}\}$ and $\{b_{n_k}\}$ are both convergent.

Hint: Use Theorem 6 twice!

Remark. Therefore from this we can conclude that **any bounded sequence in $\mathbb{R}^2$ has a convergent subsequence**. This is because any sequence in $\mathbb{R}^2$ is of the form $\{\tilde{x}_n\} = \{(a_n, b_n)\}$, and that $\{\tilde{x}_n\}$ is bounded in $\mathbb{R}^2$ means $\{a_n\}, \{b_n\}$ are bounded.

9. Let $p > 0$ and $\lim_{n\to\infty} n^p a_n = A$, including the case $A = \pm\infty$. Show that if $p > 1$ and A is finite, then $\sum_{n=1}^{\infty} a_n$ converges; if $p \le 1$ and $A \ne 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

The next two exercises will be technical.

10. (2007 Fall) Let $a_1, a_2, \cdots > 0$. Prove that if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}+a_{n+2}} = 0$, then $\{a_n\}$ is unbounded.

11. (2009 Fall) Let $a_1, a_2, \cdots > 0$ and for $n = 1, 2, \ldots$, let

$$P_n(x) = (x+1)(x+2)\cdots(x+n)$$

and

$$Q_n(x) = (x+a_1)(x+a_2)\cdots(x+a_n).$$

(a) For every $x \in \mathbb{R}$, determine whether $\sum_{n=1}^{\infty} \frac{P_n(x)}{n!} x^n$ converges or not.

(b) Prove that $\lim_{n\to\infty} \frac{a_n}{Q_n(1)} = 0$.