Math3033 (Fall 2013-2014) — Tutorial Note 4

Basic Concept and Properties of Limit Inferior and Limit Superior

Key Definitions and Results

Definition 1. Given a sequence $\{a_n\}$ of real numbers, we denote

$$\mathcal{L} = \left\{ \ell \in [-\infty,\infty] : \ell = \lim_{k\to\infty} a_{n_k} \text{ for some subsequence } \{a_{n_k}\} \text{ of } \{a_n\} \right\}$$

the collection of **subsequential limits** of $\{a_n\}$.

Definition 2. Let $a_1, a_2, \cdots \in \mathbb{R}$ and $\mathcal{L}$ the set of its subsequential limits, we denote

$$\underline{\lim}_{n\to\infty} a_n = \inf \mathcal{L} \quad \text{and} \quad \overline{\lim}_{n\to\infty} a_n = \sup \mathcal{L}$$

the **limit inferior (or lower limit)** and **limit superior (or upper limit)** respectively.

Theorem 3 (M_k & m_k). For $a_1, a_2, \cdots \in \mathbb{R}$ we have

$$\overline{\lim}_{n\to\infty} a_n = \lim_{k\to\infty} M_k = \lim_{k\to\infty} \sup\{a_k, a_{k+1}, \ldots\}$$

and

$$\underline{\lim}_{n\to\infty} a_k = \lim_{k\to\infty} m_k = \lim_{k\to\infty} \inf\{a_k, a_{k+1}, \ldots\}.$$

Theorem 4. Let $a_1, a_2, \cdots \in \mathbb{R}$ and $\mathcal{L}$ the set of its subsequentialy limits, we have:

(i) $\overline{\lim}\, a_n, \underline{\lim}\, a_n \in \mathcal{L}$.

(ii) $\{a_n\}$ converges $\iff \overline{\lim}\, a_n = \underline{\lim}\, a_n < \infty$.

(iii) $\overline{\lim}(-a_n) = -\underline{\lim}\, a_n$ (so $\underline{\lim}(-a_n) = -\overline{\lim}\, a_n$).

(iv) If $c > 0$, $\overline{\lim}(ca_n) = c\,\overline{\lim}\, a_n$ and $\underline{\lim}(ca_n) = c\,\underline{\lim}\, a_n$.

(v) If $c \in \mathbb{R}$, $\overline{\lim}(c + a_n) = c + \overline{\lim}\, a_n$ and $\underline{\lim}(c + a_n) = c + \underline{\lim}\, a_n$.

Theorem 5. If $x_n \le y_n$ for all n, then

$$\overline{\lim}\, x_n \le \overline{\lim}\, y_n \quad \text{and} \quad \underline{\lim}\, x_n \le \underline{\lim}\, y_n.$$

Theorem 6. Let $a_1, a_2, \cdots \in \mathbb{R}$, and $r \in \mathbb{R}$.

(i) If $\overline{\lim}\, a_n < r$, then $a_n < r$ for all but finitely many n.

(ii) If $r < \underline{\lim}\, a_n$, then $r < a_n$ for all but finitely many n.

Example 1. Find $\overline{\lim}$ and $\underline{\lim}$ of $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_n = \sin \frac{n\pi}{3}.$$

Solution. We observe that

$$\{x_n\} = \left\{ \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, \ldots \right\}.$$

Therefore it is easy to see that all subsequential limits are

$$\mathcal{L} = \left\{ -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2} \right\},$$

and therefore $\overline{\lim}\, x_n = \frac{\sqrt{3}}{2}$ and $\underline{\lim}\, x_n = -\frac{\sqrt{3}}{2}$.

Example 2. Find $\overline{\lim}$ and $\underline{\lim}$ of the sequence $\{a_n\}_{n=1}^{\infty}$ defined by

$$a_n = \frac{n^2}{1+n^2}\cos\frac{2n\pi}{3}.$$

Solution. We first study how a_n's oscillate:

$$\cos\frac{2n\pi}{3} = \left\{-\frac{1}{2},-\frac{1}{2},1,-\frac{1}{2},-\frac{1}{2},1,\dots\right\}.$$

Therefore it is natural to divide the sequences $\{a_n\}$ into groups of 3. Indeed, we have

$$\{a_n\} = \{-,-,+,-,-,+,-,-,+,\dots\}$$

Thus it is easy to see that by cancelling countably many negative terms,

$$M_{3n} = \sup\{a_{3n},a_{3n+3},\dots\} = \sup\left\{\frac{(3n)^2}{1+(3n)^2}\right\} = \lim_{n\to\infty}\frac{(3n)^2}{1+(3n)^2} = 1.$$

Since $\{M_n\}$ is decreasing, we have

$$\overline{\lim}\, a_n = \lim_{n\to\infty} M_n = \lim_{n\to\infty} M_{3n} = \lim_{n\to\infty} 1 = 1.$$

Next we focus on negative terms to compute $\underline{\lim}$. By cancelling countably many positive terms we have

$$m_{3n+1} = \inf\{a_{3n+1},a_{3n+2},\quad a_{3n+4},a_{3n+5},\quad \dots\}.$$

We compare

$$a_{3n+1} = \frac{(3n+1)^2}{1+(3n+1)^2}\left(-\frac{1}{2}\right) \quad\text{and}\quad a_{3n+2} = \frac{(3n+2)^2}{1+(3n+2)^2}\left(-\frac{1}{2}\right),$$

as the larger the magnitude, the smaller the value, thus we discard a_{3n+1} in each group to conclude

$$m_{3n+1} = \inf\{a_{3n+2},a_{3n+5},a_{3n+8},\dots\} = \lim_{n\to\infty}\frac{(3n+2)^2}{1+(3n+2)^2}\left(-\frac{1}{2}\right) = -\frac{1}{2}.$$

Since $\{m_n\}$ is increasing,

$$\underline{\lim}\, a_n = \lim_{n\to\infty} m_n = \lim_{n\to\infty} m_{3n+1} = -\frac{1}{2}.$$

Example 3. Prove Theorem 5:

If $x_n \le y_n$ for all n, then

$$\overline{\lim}\, x_n \le \overline{\lim}\, y_n \quad\text{and}\quad \underline{\lim}\, x_n \le \underline{\lim}\, y_n.$$

by using Theorem 6.

Solution. We first prove the inequality $\overline{\lim}\, x_n \le \overline{\lim}\, y_n$. If $\overline{\lim}\, y_n = \infty$, then we are done. Suppose now $\overline{\lim}\, y_n < \infty$. If we fix an $\epsilon > 0$, then

$$\overline{\lim}\, y_n < \boxed{\overline{\lim}\, y_n + \epsilon,}$$

hence by Theorem 6 there is an $N \in \mathbb{N}$ such that

$$n > N \implies y_n < \overline{\lim}\, y_n + \epsilon \implies x_n < \overline{\lim}\, y_n + \epsilon.$$

Recall that $\overline{\lim}\, x_n$ is a subsequential limit, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = \overline{\lim}\, x_n$. Now

$$k \text{ big} \implies x_{n_k} < \overline{\lim}\, y_n + \epsilon.$$

By taking $k \to \infty$,

$$\overline{\lim}\, x_n \le \overline{\lim}\, y_n + \epsilon.$$

Since this is true for each $\epsilon > 0$, by letting $\epsilon \to 0^+$, we have $\overline{\lim}\, x_n \le \overline{\lim}\, y_n$.

Of course the second inequality can be proved directly using the technique in the first paragraph, what we will do is: deduce the second inequality involving $\underline{\lim}$ from what we have proved. Since $-x_n \ge -y_n$, by taking $\overline{\lim}$ on both sides, we have

$$\overline{\lim}(-x_n) \ge \overline{\lim}(-y_n).$$

By (iii) of Theorem 4 we have

$$-\underline{\lim}\, x_n \ge -\underline{\lim}\, y_n,$$

therefore we have $\underline{\lim}\, x_n \le \underline{\lim}\, y_n$.

Example 4 (2004 Midterm (L1)). Let $a_k \geq 0$ for $k \geq 1$ and $\sum_{k=1}^{\infty} a_k$ converges, prove that

$$\underline{\lim}\, k a_k = 0.$$

Give an example to show in the above situation, it is possible that $\overline{\lim}\, k a_k = 1$.

Solution. There are two ways to show $\underline{\lim}\, k a_k = 0$.

Method 1. We prove by contradiction. Suppose $\underline{\lim}\, k a_k > 0$, then there is $\alpha > 0$ such that $\underline{\lim}\, k a_k > \alpha > 0$. By Theorem 6 there is a $K \in \mathbb{N}$ such that

$$k > K \implies k a_k > \alpha,$$

and this implies $a_k > \frac{\alpha}{k}$, so if we sum both sides from $K+1$ to ∞,

$$\sum_{k=K+1}^{\infty} a_k \geq \alpha \sum_{k=K+1}^{\infty} \frac{1}{k},$$

this implies $\sum_{k=1}^{\infty} a_k$ diverges, a contradiction to the hypothesis.

Method 2. Let's fix a k and let

$$m_k = \inf\{k a_k, (k+1)a_{k+1}, \dots\},$$

then for each $n \geq k$, $m_k \leq n a_n \implies \frac{m_k}{n} \leq a_n$, by summing from $n = k$ to $n = \infty$,

$$m_k \sum_{n=k}^{\infty} \frac{1}{n} \leq \sum_{n=k}^{\infty} a_n,$$

since $\sum_{n=1}^{\infty} a_n$ is finite and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, necessarily $m_k = 0$. As this holds for each k,

$$\underline{\lim}\, k a_k = \lim m_k = 0.$$

Finally, we need to raise an example that:

1) $\underline{\lim}\, k a_k = 0$: For this, we introduce infinitely many 0 to a_k's since $a_k \geq 0$.
2) $\overline{\lim}\, k a_k = 1$: For this, we introduce infinitely many 1 to $k a_k$ and keep $k a_k \leq 1$. For instance, we let infinitely many k be s.t. $a_k = 1/k$, and set $a_k = 0$ otherwise, then the $\mathcal{L}$ set can only be $\{0, 1\}$.
3) $\sum a_k < \infty$: For this, our k's s.t. $a_k = 1/k$ should be far enough from each other.

One possible choice to fulfil 2) and 3) is to let

$$a_k = \begin{cases} 1/k, & k = n^2 \text{ for some } n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Of course then 1) is also satisfied.

Exercise 1. Let $a_n \geq 0$ for $n \geq 1$, show that $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} \leq 1$ if and only if $\lim_{n\to\infty} \frac{a_n}{\ell^n} = 0$ for every $\ell > 1$.

Exercise 2 ($\frac{*}{\infty}$ form of Stolz-Cesàro Theorem). In this exercise we prove a refined version of Stolz Theorem that we have leant in Math2031:

Show that if $\{a_n\}$ and $\{b_n\}$ are two real sequences, $b_n \nearrow \infty$, then

$$\underline{\lim_{n\to\infty}} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} \leq \underline{\lim_{n\to\infty}} \frac{a_n}{b_n} \leq \overline{\lim_{n\to\infty}} \frac{a_n}{b_n} \leq \overline{\lim_{n\to\infty}} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}.$$

Exercise 3. The following are applications of Stolz Theorem.

(a) Let $\alpha > 1$, prove that

$$\lim_{n\to\infty} n \left(\frac{1^\alpha + 2^\alpha + \cdots + n^\alpha}{n^{\alpha+1}} - \frac{1}{\alpha+1} \right) = \frac{1}{2}.$$

(b) Let $a_1, a_2, \dots \in \mathbb{R}$ be such that $\lim_{n\to\infty} a_n \sum_{i=1}^{n} a_i^2 = 1$, prove that

$$\lim_{n\to\infty} \sqrt[3]{3n}\, a_n = 1.$$

(c) Prove that

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=0}^{n} \ln \binom{n}{k} = \frac{1}{2}.$$