Infimum and Supremum

We need to know

- how to find supremum and infimum in a systematical way.

Key definitions and results

Definition 1 (Lower, Upper Bound).

- A set S is **bounded below** if there is an $m \in \mathbb{R}$ such that $m \le x$ for all $x \in S$, such m is called a **lower bound** of S.
- Similarly, a set S is **bounded above** if there is an $M \in \mathbb{R}$ such that $x \le M$ for all $x \in S$, such M is called an **upper bound** of S.

Definition 2 (Infimum). If a set is bounded below, then the **infimum** of S, denoted by $\inf \boldsymbol{S}$, is a *lower bound* of S such that for any lower bound m of S, we have

$$m \le \inf S.$$

It is also called the greatest lower bound of S.

Definition 3 (Supremum). If a set is bounded above, then the **supremum** of S, denoted by $\sup \boldsymbol{S}$, is an *upper bound* of S such that for any upper bound M of S, we have

$$\sup S \le M.$$

It is also called the least upper bound of S.

Definition 4 (Convergence of Seqeucne). We say that a sequence $\{a_n\}$ **(or a_n) converges to a**, denoted by $\lim_{n\to\infty} a_n = a$ or $a_n \to a$, if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \quad \text{s.t.} \quad n > N \implies |a_n - a| < \epsilon.$$

Theorem 5 (Infimum Property). If a set S has an infimum in $\mathbb{R}$. Then for every $\epsilon > 0$, there is an $x \in S$ such that $\inf S \le x < \inf S + \epsilon$.

Theorem 6 (Infimum Limit). Let S be a nonempty set that is bounded below. Then a number $m = \inf S$ if and only if

(a) m is a lower bound.

(b) There is a sequence $\{x_n\}$ in S such that $\lim_{n\to\infty} x_n = m$.

Theorem 7 (Supremum Property). If a set S has a supremum in $\mathbb{R}$. Then for every $\epsilon > 0$, there is an $x \in S$ such that $\sup S - \epsilon < x \le \sup S$.

Theorem 8 (Supremum Limit). Let S be a nonempty set that is bounded above. Then a number $M = \sup S$ if and only if

(a) M is an upper bound.

(b) There is a sequence $\{x_n\}$ in S such that $\lim_{n\to\infty} x_n = M$.

Theorem 9 (Density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$). For every $x < y$ there is an $m/n \in \mathbb{Q}$ and also a $w \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$x < \frac{m}{n} < y \quad \text{and} \quad x < w < y.$$

Example 1. Determine if each of the following sets has an infimum and a supremum in $\mathbb{R}$. If they exist, find them and explain.

(a) $A = \left\{ \dfrac{\sqrt{2}}{m+n} + \dfrac{1}{k\sqrt{2}} : m,n,k \in \mathbb{N} \right\}$. (Practice Exercise 91(g))

(b) $B = \left\{ \dfrac{k}{n!} : k,n \in \mathbb{N}, \dfrac{k}{n!} < \sqrt{2} \right\}$. (Practice Exercise 91(m))

(c) $C = \{a+b : a,b \in \mathbb{Q}, a^2 < 3, |2b+1| < 5\}$.

Sol **(c) Proof of** $\inf C = -\sqrt{3} - 3$**.** We divide this part into two steps.

Step 1. Find a suitable lower bound of C.
For every $x \in C$, there are $a,b \in \mathbb{R}$ such that $a^2 < 3$ and $|2b+1| < 5$ and

$$x = a + b.$$

Note that

$a^2 < 3 \iff |a| < \sqrt{3} \iff -\sqrt{3} < a < \sqrt{3}$ and
$|2b+1| < 5 \iff -5 < 2b+1 < 5 \iff -3 < b < 2$,

therefore we have

$$-\sqrt{3} - 3 < x = a + b < \sqrt{3} + 2. \qquad (*)$$

This holds for every $x \in C$, we conclude C is bounded below by $-\sqrt{3} - 3$.

Step 2. Find a sequence in C that converges to this lower bound.
Let's choose $a_n = -\sqrt{3} + \frac{1}{n}$ and $b_n = -3 + \frac{1}{n}$. Then $a_n^2 < 3$ and $|2b_n + 1| < 5$, so the number

$$C \ni a_n + b_n = -\sqrt{3} - 3 + \frac{2}{n} \to -\sqrt{3} - 3.$$

By Infimum Limit Theorem, $\inf C = -\sqrt{3} - 3$.

Proof of $\sup C = \sqrt{3} + 2$**.** This is similar to the above.

By $(*)$ $\sqrt{3} + 2$ is an upper bound of C.

The sequence $a_n = \sqrt{3} - \frac{1}{n}$, $b_n = 2 - \frac{1}{n}$ satisfies

$$C \ni a_n + b_n \to \sqrt{3} + 2.$$

Therefore $\sup C = \sqrt{3} + 2$. ∎

Example 2. Let A and B be bounded.

(a) Let $A + B = \{a + b : a \in A, b \in B\}$. Show that

$$\sup(A+B) = \sup A + \sup B$$

and

$$\inf(X+Y) = \inf X + \inf Y.$$

(b) Let $cX = \{cx : x \in X\}$. Show that

$$\sup(cX) = c \sup X \quad \text{when } c > 0$$

and

$$\sup(cX) = c \inf X \quad \text{when } c < 0.$$

Sol **(a)** For every $x \in A + B$, there are $a \in A, b \in B$, such that $x = a + b \le \sup A + \sup B$, so $\sup A + \sup B$ is an upper bound of $A + B$.

We construct a sequence in $A + B$ that converges to $\sup A + \sup B$. By Supremum Limit Theorem, there are sequences $\{a_n\}$ in A and $\{b_n\}$ in B such that

$$a_n \to \sup A \quad \text{and} \quad b_n \to \sup B.$$

Therefore

$$A + B \ni a_n + b_n \to \sup A + \sup B.$$

We conclude that $\sup(A+B) = \sup A + \sup B$.

The rest of part (a) and also part (b) are similar. ∎

Example 3. Find $\sup A$ and $\inf A$, where

$$A=\left\{\frac{m}{\sqrt{3}\times 2^n}: m\in\mathbb{Z}, n\in\mathbb{N}\right\}\cap(0,4).$$

Also define $S=\left\{y-\frac{1}{e^x}: x,y\in A\right\}$, what is $\sup S$ and $\inf S$?

Sol **(i)** By definition $x>0$ for every $x\in A$. Since

$$A\ni\frac{1}{\sqrt{3}\times 2^n}\to 0,$$

therefore $\inf A=0$.

(ii) Now we show that $\sup A=4$.

4 is an upper bound of A by definition.

Since for every $y\in\mathbb{R}$ we have $[y\cdot 2^n]/2^n\to y$, where $[y]$ denotes the integral part of y. Therefore

$$A\ni\frac{[4\sqrt{3}\cdot 2^n]}{\sqrt{3}\cdot 2^n}\to\frac{4\sqrt{3}}{\sqrt{3}}=4.$$

We conclude $\sup A=4$.

Simple manipulation of inequalities gives

$$\inf S=-1\quad\text{and}\quad\sup S=4-\frac{1}{e^4}.$$

For example, let's prove $\sup S=4-1/e^4$ rigorously, the fact that $\inf S=-1$ is left to you.

Indeed, we see that for every $x,y\in A$, $\inf A\le x,y\le\sup A$, therefore

$$y-\frac{1}{e^x}\le\sup A-\frac{1}{e^{\sup A}}=4-\frac{1}{e^4},$$

so $4-\frac{1}{e^4}$ is an upper bound of S. Next, by Supremum Limit Theorem, there is a sequence $A\ni x_n\to 4$, and thus

$$S\ni x_n-\frac{1}{e^{x_n}}\to 4-\frac{1}{e^4},$$

and therefore $\boxed{\sup S=4-\frac{1}{e^4}}$. ∎

Exercises

1. (2006 Fall Midterm) Let $\left(0,\frac{1}{2}\right)\cap\mathbb{Q}\subseteq A_1\subseteq[0,1)$. For $n=1,2,\ldots$ we let

$$A_{n+1}=\{\sqrt{x}: x\in A_n\}.$$

Determine the supremum and infimum of $\bigcup_{k=1}^{\infty}A_k$ with proof.

2. (2005 Final) Determine the supremum of

$$S=\bigcup_{n=1}^{\infty}\left\{\frac{1}{x}+\frac{1}{n\sqrt{2}}: x\in(2,3]\setminus\mathbb{Q}\right\}$$

and be sure to give a proof for your answer.

3. (2002 Spring) Suppose $\{x_n\}$ converges to $w\in\mathbb{R}$ and $x_n<w$ for all $n\in\mathbb{N}$. Now for each $n\in\mathbb{N}$ we let

$$y_n=\sup\left\{x_{2k}: k\in\mathbb{N}, k\le\frac{n+1}{2}\right\}.$$

Show that $\{y_n\}$ converges to w.

4. Find supremum and infimum of each of the following sets:

(a) $\{\sqrt{n}-[\sqrt{n}]: n\in\mathbb{N}\}$(*);

(b) $\left\{\frac{\alpha m+\beta n}{m+n}: m,n\in\mathbb{N}, m+n\neq 0\right\}$, $\alpha,\beta>0$;

(c) $\left\{\frac{m^2-n}{m^2+n^2}: n,m\in\mathbb{N}, m>n\right\}$;

(d) $\left\{\frac{n-k^2}{n^2+k^3}: n,k\in\mathbb{N}\right\}$.

(*) $[x]$ denotes the **integral part** of x, which is the *biggest* integer not exceeding x. e.g., $[2.033]=2$ and $[-1.033]=-2$. It is also commonly denoted by $\lfloor x\rfloor$ $(=[x])$, called **floor function**.