Math2033 Mathematical Analysis (Spring 2013-2014) Tutorial Note 2

Countability

We need to know

- how to judge whether a set is countable;
- how to prove the existence of some desired element via countability.

Key definitions and results

Definition 1 (Countability).

- A set S is **countably infinite** if there is a bijection $f : \mathbb{N} \to S$.
- A set is said to be **countable** if it is finite or countably infinite.
- A set is said to be **uncountable** if it is not countable.

Theorem 2 (Countable Subset). Suppose $A \subset B$, then:

(a) B is countable $\implies$ A is countable.

(b) A is uncountable $\implies$ B is uncountable.

Theorem 3 (Bijection). If there is a bijection $f : A \to B$, then

$$A \text{ is countable} \iff B \text{ is countable.}$$

Theorem 4 (Countable Union).

$$S \text{ is countable and } A_s \text{ is countable } \forall s \in S \implies \bigcup_{s\in S} A_s \text{ is countable.}$$

Theorem 5 (Product).

$$A_1, A_2, \ldots, A_n \text{ are countable} \implies A_1 \times A_2 \times \cdots \times A_n \text{ is countable.}$$

Theorem 6 (Injection). Let $f : A \to B$ be injective, then

$$B \text{ is countable} \implies A \text{ is countable.}$$

Theorem 7 (Surjection). Let $g : A \to B$ be surjective, then

$$A \text{ is countable} \implies B \text{ is countable.}$$

Remark. We cannot replace n by ∞ in Product Theorem 5, a textbook example is

$$\{0,1\}^{\infty} := \{0,1\} \times \{0,1\} \times \cdots$$

which is uncountable.

Example 1. Determine whether the following set is countable:

$$A := \{x \in \mathbb{R} : 9\sin^9 x + 3\sin^3 x + 1 = 0\}$$

Sol The usual technique is to rewrite the expression of A. To do this, we find a equivalent statement of "$a \in A$".

Note that

$$\begin{aligned}
& a \in A \\
\iff & 9\sin^9 a + 3\sin^3 a + 1 = 0 \\
\iff & \sin a \text{ is a zero (i.e., root) of } P(x) := 9x^9 + 3x^3 + 1 \\
\iff & \sin a \in Z(P) := \{x \in \mathbb{R} \in \mathbb{R} : P(x) = 0\} \\
\iff & \sin a = k \text{ for some } k \in Z(P) \\
\iff & a \in \{x \in \mathbb{R} : \sin x = k, \exists k \in Z(P)\} \\
\iff & a \in \bigcup_{k\in Z(P)} \{x \in \mathbb{R} : \sin x = k\} \\
\iff & a \in \bigcup_{k\in Z(P)} \{x \in \mathbb{R} : x = n\pi + (-1)^n \sin^{-1} k, \exists n \in \mathbb{Z}\} \\
\iff & a \in \bigcup_{k\in Z(P)} \bigcup_{n\in\mathbb{Z}} \{x \in \mathbb{R} : x = n\pi + (-1)^n \sin^{-1} k\} \\
\iff & a \in \bigcup_{k\in Z(P)} \bigcup_{n\in\mathbb{Z}} \{n\pi + (-1)^n \sin^{-1} k\}.
\end{aligned}$$

Since every **nonzero** has at most finitely many zeros, $Z(P)$ is necessarily a finite set, hence a countable set. Next $\mathbb{Z}$ is countable, and therefore by Countable Union Theorem twice,

$$A = \bigcup_{k\in Z(P)} \bigcup_{n\in\mathbb{Z}} \{n\pi + (-1)^n \sin^{-1} k\}$$

is countable. ∎

Example 2 (Practice Exercise 23). Determine if the set E of all circles in $\mathbb{R}^2$ with centers at rational coordinate points and positive rational radius is countable.

Sol Let $C(\vec{x}, r)$ denote the circle with center $\vec{x} \in \mathbb{R}^2$ and radius $r \geq 0$. Also let's denote $\mathbb{Q}^+ = \mathbb{Q} \cap (0, \infty)$.

We need to be careful what is the object we are going to count. Of course any circle, being a subset of $\mathbb{R}^2$, must be uncountable. But would we say a human being, an entity, is uncountable just because it is made up of a myriad of biological cells?

Our object to count is *circles*.

By definition, $E = \left\{C\left((a,b), r\right) : a, b \in \mathbb{Q}, r \in \mathbb{Q}^+\right\}$, and therefore

$$E = \bigcup_{a\in\mathbb{Q}} \bigcup_{b\in\mathbb{Q}} \bigcup_{r\in\mathbb{Q}^+} \left\{(C((a,b),r)\right\},$$

by Countable Union Theorem thrice, E is countable. ∎

Example 3 (Practice Exercise 89). Determine if the following sets are countable or not.

(a) S is the set of all intersection points (x, y) of the line $y = \pi x$ with the graphs of all equations $y = x^3 + x + m$, $m \in \mathbb{Z}$.

(i) $S = \{x^2 + y^2 : x, y \in A\}$, where A is a nonempty countable subset of $\mathbb{R}$.

Sol (a) $\vec{a} \in S$

$\iff \vec{a} \in \overbrace{\{(x,y) : y = \pi x\} \cap \{(x,y) : y = x^3 + x + m\}}^{:=I_m}$, for some $M \in \mathbb{Z}$
$\iff \vec{a} \in \bigcup_{m\in\mathbb{Z}} I_m$,

so $S = \bigcup_{m\in\mathbb{Z}} I_m$.

Note that I_m has at most 3 points, since if we solve the the intersection, then $(x, y) = (x, \pi x)$ lies on $y = x^3 + x + m$ iff $\pi x = x^3 + x + m$, which has at most 3 solutions.

Thus S is countable by Countable Union Theorem once.

(i) $\vec{a} \in S$ iff $\vec{a} = x^2 + y^2$ $\exists x \in A, \exists y \in Y$ iff $a \in \bigcup_{x\in A} \bigcup_{y\in A} \{x^2 + y^2\}$, Countable Union Theorem twice will do. ∎

Example 4. Show that the **power set** of $\mathbb{N}$, denoted by $2^{\mathbb{N}}$ or $\mathcal{P}(\mathbb{N})$, is an uncountable set.

Sol Recall the power set of X is defined to be the collection of all possible subsets of X. For example, if $X = \{1, 2, 3\}$, then

$$2^{\{1,2,3\}} = \left\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{3,1\}, \{1,2,3\}\right\}.$$

The number of elements in $2^{\{1,2,3\}}$ is $2^3 = 8$.

In general, if $|X| = n$, then $|2^X| = 2^n$, this motivate the notation 2^X.

To show $2^{\mathbb{N}}$ is uncountable, we try to show it is bijective to an uncountable set.

To form a subset of $\mathbb{N}$, say A, for each $k \in \mathbb{N}$ we either put k into A or cast it aside. If we choose k, we assign k a value 1, if we abandon it, we assign k a value 0.

Graphically, we have

1	2	3	4	5	6	7	8	9	10	...
|	|	|	|	|	|	|	|	|	|	
0	1	1	0	1	0	0	1	0	0	

to mean we choose $2, 3, 5, 8, \ldots$ and don't choose the remaining which is assigned a value 0. Therefore it is easy to imagine now

$$2^{\mathbb{N}} \longleftrightarrow \{(a_1, a_2, \ldots) : a_1, a_2, \cdots \in \{0,1\}\} = \{0,1\}^{\infty}.$$

In lecture we know that $\{0,1\}^{\infty}$ is uncountable, thus we are done. ∎

Remark. We can also state a much general fact in terms of **Cardinality**. In Exercise 6 if we replace X by $\mathbb{N}$ we have $|2^{\mathbb{N}}| > |\mathbb{N}|$, therefore $2^{\mathbb{N}}$ is uncountable.

Remark. The same technique to construct bijection from a set of functions on $\mathbb{N}$ to a set of sequences can be used to solve Exercise 3, try to practice more!

Example 5. A real number $a \in \mathbb{R}$ is said to be **algebraic** if there is a **nonzero** polynomial $P(x) \in \mathbb{Q}[x]$ such that $P(a) = 0$.

(a) Show that the set of algebraic numbers is countable.

(b) A number is said to be **transcendental** if it is not algebraic. Explain why there must be a transcendental number.

Sol **(a)** To understand the countability of the set of algebraic numbers, we need to rewrite the expression:

a is algebraic
iff $P(a) = 0$ for some $P \in \mathbb{Q}[x] \setminus \{0\}$ (definition)
iff $a \in Z(P)$ for some $P \in \mathbb{Q}[x] \setminus \{0\}$
(definition, recall $Z(P) := \{x \in \mathbb{R} : P(x) = 0\}$)
iff $a \in \bigcup_{P \in \mathbb{Q}[x]\setminus\{0\}} Z(P)$.

The iff's mean

$$\{\text{algebraic number}\} = \bigcup_{P \in \mathbb{Q}[x]\setminus\{0\}} Z(P). \qquad (*)$$

Now $Z(P)$ is just a finite set (hence countable) since P, as a **nonzero** polynomial, has only finitely many zeros.

In view of Countable Union Theorem, it is enough to show $\mathbb{Q}[x]$ is countable.

To see this, rewrite this as

$$\begin{aligned}\mathbb{Q}[x] &= \bigcup_{n=0}^{\infty} \{a_0 + a_1 x + \cdots + a_n x^n : a_0, a_1, \ldots, a_n \in \mathbb{Q}\} \\ &= \bigcup_{n=0}^{\infty} \bigcup_{(a_0, a_1, \ldots, a_n) \in \mathbb{Q}\times\mathbb{Q}\times\cdots\times\mathbb{Q}} \{a_0 + a_1 x + \cdots + a_n x^n\}\end{aligned}$$

Here $\mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}$ is countable by Product Theorem, therefore $\mathbb{Q}[x]$ is countable by Countable Union Theorem.

Finally the set in $(*)$ is countable again by Countable Union Theorem.

(b) The existence of transcendental number is now obvious because

$$\mathbb{R} \setminus \{\text{algebraic number}\}$$

is uncountable, therefore nonempty. ∎

Exercises

1. Determine with proof whether each of the following sets is countable:

(a) $A = \{\sqrt{|x| + y} : x \in \mathbb{Z}, y \in (0,1) \setminus \mathbb{Q}\}$
(b) $B = \{\sin x + \cos y : x, y \in \mathbb{R}\}$
(c) $C = \left\{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\right\}$
(d) $D = \{[x]^2 + y : x \in \mathbb{R}, y \in \mathbb{Q}\}$, $[x]$ is the largest integer not exceeding x

2. Show that the set $\mathcal{F}$ of all finite subsets of $\mathbb{N}$ is countable.

Caution: Note that $\mathcal{F} \subseteq 2^{\mathbb{N}}$ and $2^{\mathbb{N}}$ was shown to be uncountable in Example 4, therefore Countable Subset Theorem is not available.

3. Is the set $\mathcal{F}$ of functions from $\mathbb{N}$ to $\{1,2,3\}$ countable? [**Insturction**: Try to show that $\mathcal{G} : \mathcal{F} \to \{1,2,3\}^\infty$ given by $\mathcal{G}(f) = (f(1), f(2), \ldots) \in \{1,2,3\}^\infty$ is bijective. Next, show that $\{1,2,3\}^\infty$ contains an uncountable subset.]

4. **(2002 Spring)** Let S be the set of all lines ℓ on the $\mathbb{R}^2$ such that ℓ passes through two distinct points in $\mathbb{Q} \times \mathbb{Q}$. Let T be the set of all points, each of which is the intersection of a pair of distinct lines in S. Determine if T is countable or not.

5. **(2003 Spring)** Let P be a countable set of points in $\mathbb{R}^2$. Prove that there exists a circle C with the origin as center and positive radius such that every point of the circle C is not in P. (Note points inside the circle do not belong to the circle)

6. **(Cardinality)** For a set A, the symbol $|A|$ is called the **cardinality** of A. It is defined to be the number of elements in A when A has just finitely many elements (i.e., a finite set). When A is an infinite set, we denote $|A| = \infty$. Such ∞'s can still be compared by further defining the following formal inequality: Let X, Y be two sets.

We say that $|X| \le |Y|$ if there is an injection from X into Y.

Therefore, for example, if $X \subseteq Y$, then $|X| \le |Y|$ since $x \mapsto x : X \to Y$ is one of possible injections. We also define strict inequality as follows:

Let X, Y be two sets, we say that $|X| < |Y|$
if there is an injection but no surjection from X into Y.

Finally we define $|X| = |Y|$ if there is a bijection between them. *Schröder-Bernstein Theorem* tells us $|X| = |Y|$ if and only if $|X| \le |Y|$ and $|X| \ge |Y|$.

Now let X be any set.

(a) Show that $|2^X| \ge |X|$.
(b) Prove that $|2^X| > |X|$ by showing there is no surjection $f : X \to 2^X$. [**Instruction**: Suppose such f exists, consider $A_f = \{x \in X : x \notin f(x)\}$ (possibly $= \emptyset$), then what happens after choosing $y \in X$ s.t. $f(y) = A_f$?]