Review of Linear Algebra and Multivariable Calculus

Key Definitions and Results

Remark 1 (Abuse of Notation). Unlike linear algebra, in analysis **column vectors and row vectors are considered the same**. Every element in $\mathbb{R}^n$ will be written as a column or a row interchangeably. We need to adjust the interpretation of a notation ourselves to make sure everything makes sense.

Definition 2. A function $T : \mathbb{R}^m \to \mathbb{R}^n$ is said to be a **linear transformation** if for every $a \in \mathbb{R}$ and every $x, y \in \mathbb{R}^m$, one has

$$T(ax) = aT(x) \quad \text{and} \quad T(x+y) = T(x) + T(y).$$

Definition 3. Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$, we say that $F : X \to Y$ is **continuous at** $\boldsymbol{a} \in X$ if for every given $\epsilon > 0$, there is a $\delta > 0$ such that for every $x \in X$,

$$\|x - a\| < \delta \implies \|F(x) - F(a)\| < \epsilon.$$

Definition 4. We say that a function f has some property **near** $\boldsymbol{a \in \mathbb{R}^n}$ if such property holds for each point of $B(a,\delta) := \{x : \|x - a\| < \delta\}$, for some $\delta > 0$.

Definition 5 (1st Course Version). Let f be a real-valued function defined near $a \in \mathbb{R}^n$, then f is said to be **differentiable at** $\boldsymbol{a}$ if there is a "tangent plane" at a such that

$$\lim_{\|x-a\|\to 0} \frac{|f(x) - (f(a) + \nabla f(a)\cdot(x-a))|}{\|x-a\|} = 0.$$

Definition 6. Let $f(x_1, x_2, \ldots, x_n)$ be a real-valued function defined near $a \in \mathbb{R}^n$ and $\{e_1, e_2, \ldots, e_n\}$ a usual basis of $\mathbb{R}^n$. For each i we define

$$\frac{\partial f}{\partial x_i}(a) = \lim_{\mathbb{R}\ni h\to 0} \frac{f(a+he_i) - f(a)}{h} =: \frac{d}{dx_i} f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n)\bigg|_{x_i = a_i}.$$

Theorem 7 (Equation of Tangent Plane).

- Suppose $f : \mathbb{R}^3 \to \mathbb{R}$. The equation of the tangent plane to the surface $f = 0$ at $a \in \mathbb{R}^3$ (s.t. $f(a) = 0$) is $\boldsymbol{\nabla f(a)\cdot(x-a) = 0}$.
- Suppose $f : \mathbb{R}^2 \to \mathbb{R}$. The equation of the tangent plane to the surface $\{(x, y, f(x,y)) : x, y \in \mathbb{R}\}$ at $(a, f(a))$ is $\boldsymbol{z = f(a) + \nabla f(a)\cdot((x,y) - a)}$.

Example 1 (Review Problem 1). For each part, find $\dfrac{dy}{dx}$ at $(1,1)$ if x and y satisfy the equation

(a) $x^4 + xy^3 + e = 2 + e^x$

(b) $x = \dfrac{e^{2y} - y^e}{e^2 - 1}$

(c) $x^7 + x^2y^2 + y^7 + e^y = 3 + e$

Must you do implicit differentiation to find $\dfrac{dy}{dx}$ in each part?

Solution. (a) $y' = (e-5)/3$.

(b) $y' = (e-1)/(2e^2 - e)$

(c) $y' = -9/(9+e)$

It is not necessary to do implicit differentiation in each part.

In (a) we can make y the subject, namely,

$$y = \sqrt[3]{\frac{5 + e^x - x^4}{x}}.$$

In (b) we can use the formula

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Finally in (c) it is seemingly hopeless to express y as a function of x, thus we must assume y is a function of x to do implicit differentiation. The feasibility will be guaranteed by the theorem in later lectures called Implicit Function Theorem. Implicit Function Theorem guarantees there exists a function $f(x)$ such that $y = f(x)$ is a solution of the equation

$$x^7 + x^2y^2 + y^7 + e^y = 3 + e.$$

Note that although it exists, there is no general method to find $f(x)$. The situation is the same as that we know the maximum of $f : [1,2] \to \mathbb{R}$ given by

$$f(x) = \frac{e^x \sin x}{\sqrt{x}} + \cos e^{e^x}$$

must occur somewhere, say $x_0 \in [1,2]$, but there is no general method to find x_0.

Example 2 (Review Problem 2).

(a) For every $(a,b) \in \mathbb{R}^2$, find $\dfrac{\partial F}{\partial x}(a,b)$ if $F(x,y) = \sin|x^2+y|$.

(b) For every $(a,b) \in \mathbb{R}^2$, find $\dfrac{\partial G}{\partial y}(a,b)$ if $G(x,y) = \begin{cases} \dfrac{x^2-y^2}{x-y} & \text{if } x \neq y \\ 3x & \text{if } x = y \end{cases}$.

Solution. (a) We consider case by case:

Case 1. When $a^2+b>0$, $F_x(a,b)$ exists with $F_x(a,b) = 2a\cos(a^2+b)$.

Case 2. When $a^2+b<0$, $F_x(a,b)$ exists with $F_x(a,b) = -2a\cos(a^2+b)$.

Case 3. If $a^2+b=0$, then we have

$$\begin{aligned} F_x(a,b) &= \lim_{h\to 0} \frac{F(a+h,b)-F(a,b)}{h} \\ &= \lim_{h\to 0} \frac{\sin(|h||2a+h|)}{h} \\ &= \begin{cases} 0, & \text{when } a=0, \\ \text{does not exist}, & \text{when } a>0, \\ \text{does not exist}, & \text{when } a<0. \end{cases} \end{aligned}$$

So in this case the only point that has derivative is $(0,0)$, i.e., $F_x(0,0)=0$.

(b) If $a \neq b$, then for (x,y) near (a,b), we still have $x \neq y$, hence $G(x,y) = x+y$, it follows that

$$G_y(x,y) = 1 \implies G_y(a,b) = 1.$$

If $a=b$, then by definition we have

$$G_x(a,b) = \lim_{h\to 0} \frac{G(a,b+h)-G(a,b)}{h} = \lim_{h\to 0} \frac{h-a}{h} = \begin{cases} 1, & a=0, \\ \text{does not exist}, & a \neq 0. \end{cases}$$

Exercise 1. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

(a) Show that f is not continuous at $(0,0)$.

(b) Compute $f_x(0,0)$ and $f_y(0,0)$.

(c) Convince yourself this function is an example that: Although f satisfies **(i)** $x \mapsto f(x,y_0)$ is continuous for each fixed $y_0 \in \mathbb{R}$; and **(ii)** $y \mapsto f(x_0,y)$ is continuous for each fixed $x_0 \in \mathbb{R}$. Yet f is not continuous.

Exercise 2. Let $G : \mathbb{R}^2 \to \mathbb{R}$ be defined by $G(x,y) = e^{|x-y|}$, compute, if exists, $G_x(a,b)$ for all $(a,b) \in \mathbb{R}^2$.

Example 3 (Review Problem 3).

(a) Find the equation of the tangent plane to the graph of the equation

$$e^x + y^2 z + z\cos x = 1$$

at (0,0,0) in $\mathbb{R}^3$.

(b) Find the equation of the tangent plane to the graph of the function

$$f(x,y) = xe^y + ye^x$$

at (0,0,0) in $\mathbb{R}^3$.

Solution. (a) The equation of the tangent plane is given by

$$\nabla(e^x + y^2 z + z\cos x - 1)(0,0,0) \cdot ((x,y,z) - (0,0,0)) = 0,$$

a simple computation gives $\nabla(e^x + y^2 z + z\cos x - 1)(0,0,0) = (1,0,1)$, therefore the equation that characterizes the tangent plane is

$$x + z = 0.$$

(b) The equation of tangent plane is directly given by the linear approximation of f at (0,0), namely,

$$z = f(0,0) + \nabla f(0,0)((x,y) - (0,0)) = 0 + (1,1)\cdot(x,y) = x + y.$$

Example 4 (Review Problem 4). Is the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T(x,y) = \left(4x - 3y, y + \frac{1}{2}x, -(x+y)\right)$$

a linear transformation? If so, what is its matrix w.r.t. the usual bases of $\mathbb{R}^2$ and $\mathbb{R}^3$? Is it injective? Is it surjective? Is it bijective?

Solution. (i) Yes, it is linear since

$$T(x,y) = \begin{bmatrix} 4 & -3 \\ 1/2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

and from linear algebra we know that $x \mapsto Ax$ is always linear when A is a matrix.

(ii) This is just the standard matrix, denoted by $[T]$, i.e.,

$$[T] = \begin{bmatrix} 4 & -3 \\ 1/2 & 1 \\ -1 & -1 \end{bmatrix}.$$

(iii) Let $S : V \to W$ be a linear map between vector spaces. Recall that the following are equivalent:

- S is injective (or 1-1);
- $\ker S = \{0\}$;
- $Sx = 0 \implies x = 0$.

We will use the third condition. Suppose that $T(x,y) = 0$, i.e.,

$$\begin{aligned} 4x - 3y &= 0 \\ \frac{1}{2}x + y &= 0 \\ -x - y &= 0. \end{aligned}$$

The first and third equations imply $x = y = 0$, thus T is injective.

(iv) By Rank-Nullity Theorem we have

$$2 = \dim\ker T + \dim\operatorname{range} T,$$

as T is injective, $\ker T = \{0\}$, so it has zero dimension, and we have $\dim\operatorname{range} T = 2$, hence $\operatorname{range} T \neq \mathbb{R}^3$, i.e., T is not surjective.

(v) Since it is not surjective, it is not bijective.

Example 5 (Review Problem 5). Let $a,b \in \mathbb{R}$ be constants and let the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T(x,y) = (ax+by, aby+3x).$$

For what values of a and b will the linear transformation be invertible?

Solution. Let $[T]$ denote its standard matrix, then

$$[T] = \begin{bmatrix} a & b \\ 3 & ab \end{bmatrix},$$

and we learn that T is invertible if and only if $[T]$ does, and $[T]$ is invertible iff $\det[T] = b(a^2-3) \neq 0$, thus $b \neq 0$, $a \neq \sqrt{3}$ and $a \neq -\sqrt{3}$.

***Example 1.** Let $A = [a_{ij}]_{3\times 3}$ be a real symmetric matrix. By using knowledge from Math2121 and Math2021:

(a) Prove that

$$\int_{\mathbb{R}^3} \exp\left(-\tfrac{1}{2}\textstyle\sum_{i,j=1}^3 a_{ij}x_ix_j\right) dV < \infty$$

if and only if A is positive definite$^{(*)}$.

(b) **(Left as Exercise)** When A is positive definite, for $b_1, b_2, b_3 \in \mathbb{R}$ show that

$$\int_{\mathbb{R}^3} \exp\left(-\tfrac{1}{2}\textstyle\sum_{i,j=1}^3 a_{ij}x_ix_j + \sum_{i=1}^3 b_ix_i\right) dV = \frac{\exp\left(\frac{1}{2}b^TA^{-1}b\right)(2\pi)^{3/2}}{\sqrt{\det A}},$$

where $b = (b_1,b_2,b_3)^T$.

Solution. (a) For clarity all vectors will be column vectors. We can write the integral as

$$I = \int_{\mathbb{R}^3} \exp(-\frac{1}{2}x^TAx)\,dV(x),$$

where $x \in \mathbb{R}^3$. Since A is symmetric, it is orthogonally diagonalizable, namely, there is an orthogonal 3×3 matrix P (i.e., $P^TP = I$) such that

$$P^TAP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix},$$

for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. Therefore by change of variable $x = Py$ we have

$$I := \int_{\mathbb{R}^3} \exp\left(-\frac{1}{2}(Py)^TAPy\right)\underbrace{|\det P|}_{=1}\,dV = \int_{\mathbb{R}^3} e^{-0.5\lambda_1y_1^2-0.5\lambda_2y_2^2-0.5\lambda_3y_3^2}\,dV.$$

By using the fact that volume integral can be computed by writing $dV = dy_1dy_2dy_3$, we have

$$I = \int_{\mathbb{R}} e^{-\frac{\lambda_1}{2}t^2}\,dt \int_{\mathbb{R}} e^{-\frac{\lambda_2}{2}t^2}\,dt \int_{\mathbb{R}} e^{-\frac{\lambda_3}{2}t^2}\,dt.$$

Each of the three integrals are at least positive, and thus we have $I < \infty$ if and only if $\lambda_i > 0$ for all i if and only if A is positive definite (given that A is symmetric).

Exercise 3. Finish part (b) of Example 1. Part (a) and (b) of Example 1 can be generalized to $\mathbb{R}^n$ once we understand how to define volume there.