Math2033 Mathematical Analysis (Spring 2013-2014) **Tutorial Note 1**

Informal Logic, Indirect Proof, Sets and Functions

We need to know

- how to negate a statement in order to conduct indirect proof;
- what are the standard set operations;
- given two sets A, B, how to judge $A = B$ or $A \subseteq B$, or other possibility;
- what do we mean by *injective* (or *one-one*), *surjective* (or *onto*) and *bijective*.

Key definitions and results

Definition 1 (Terminology in Logic).

- A **statement/proposition**, usually denoted p or q, is a sentence *with truth value* (i.e., it is either true or false).
- Let p be a statement, its **negation**—NOT p—is denoted by $\sim p$.
- A **conditional statement** is of the form *if p then q*, denoted by $p \implies q$.
- We say that p **if and only if** q if we have $p \implies q$ and $q \implies p$ at the same time. In that case, we also say p **iff** q, $p \iff q$ and p & q are **equivalent**.
- In definitions, **if** is actually **if and only if**.
- Given a conditional statement $p \implies q$, its **contrapositive** is the following *equivalent* conditional statement:

$$\sim q \implies \sim p$$

- Some statements consist of the following two **Quantifiers**:

  Turned A: $\boxed{\forall}$ denotes for all, for each, for every;

  Turned E: $\boxed{\exists}$ denotes for some, there is (at least one), there are (some).

Definition 2 (Standard Set Operations, Notations). Let A and B be two sets.

- $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- $A \setminus B = \{x : x \in A, x \notin B\}$
- $A \times B = \{(a,b) : a \in A, b \in B\}$
- $\mathbb{R} = (-\infty, \infty)$
- $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$
- $\mathbb{Q} = \{x \in \mathbb{R} : x \text{ rational}\}$
- $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$
- $\mathbb{N} = \{1, 2, 3, \dots\}$
- More generally, given sets $A_1, A_2, \dots$ we denote

$$\bigcup_{i=1}^{\infty} A_i := A_1 \cup A_2 \cup \cdots \quad \text{and} \quad \bigcap_{i=1}^{\infty} A_i := A_1 \cap A_2 \cap \cdots.$$

  The notations $\bigcup_{i=1}^{n}$ and $\bigcap_{i=1}^{n}$ are similarly defined.

- The **Cartesian product** $A \times B$ can be conducted infinitely many times:

$$\prod_{i=1}^{\infty} A_i := A_1 \times A_2 \times \cdots = \{(a_1, a_2, \dots) : a_1 \in A_1, a_2 \in A_2, \dots\}$$

Definition 3 (1-1, onto, bijective). Let $f : X \to Y$ be a function between two sets. We say:

- f is **injective/one-one** if $f(x) = f(y) \implies x = y$.
- f is **surjective/onto** if for every $y \in Y$, there is an $x \in X$, $f(x) = y$.
- f is **bijective** if it is both injective and surjective.

Definition 4 (Set's $\subseteq$, $=$). Let A, B be two sets, we say that $\boldsymbol{A \subseteq B}$ if there holds $(\forall x)\ x \in A \implies x \in B$. Moreover, we say that $\boldsymbol{A = B}$ if $A \subseteq B$ and $B \subseteq A$.

Theorem 5 (Negation). Given statements p and q, we have the following rules:

- $\sim(\forall x, \exists y, S(x,y)) = \exists x, \forall y, \sim S(x,y)$
- $\sim(\exists x, \forall y, S(x,y)) = \forall x, \exists y, \sim S(x,y)$
- $\sim(\sim p) = p$
- $\sim(p \text{ and } q) = \sim p \text{ or } \sim q$
- $\sim(p \text{ or } q) = \sim p \text{ and } \sim q$
- $\sim(p \implies q) = p \text{ and } \sim q$

Example 1. Negate the following statements:

(a) $A \subseteq B$.

(b) $\forall \epsilon > 0, \exists \delta > 0, \qquad |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$.

(c) $\forall \epsilon > 0, \exists N \in \mathbb{N}, \qquad n > N \implies |a_n - a| < \epsilon$.

(d) $\exists M > 0, \exists N \in \mathbb{N}, \qquad n > N \implies |a_n| < M$.

Sol (a) $A \subseteq B$ is the same as $\forall x \in A$, $x \in B$, therefore the negation is

$$\exists x \in A, x \notin B.$$

(b) The precise definition of "$\implies$" in the statement above must be reinterpreted as

$$\forall \epsilon > 0, \exists \delta > 0, \boldsymbol{\forall x}, \quad |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon.$$

Now the negation is

$$\exists \epsilon > 0, \forall \delta > 0, \boldsymbol{\exists x}, \quad |x - x_0| < \delta \quad \text{and} \quad |f(x) - f(x_0)| \geq \epsilon.$$

(c) and (d) are the same as (b). ∎

Example 2. Write down the contrapositive of the following known results:

(a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

(b) If $f(x)$ is differentiable at a, then it is continuous at a.

Sol (a) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

(b) If $f(x)$ is not continuous at a, then it is not differentiable at a. ∎

Example 3. Show that $\sqrt{2}$ is an irrational number.

Sol We prove by contradiction, suppose $\sqrt{2}$ is rational, then there are $a, b \in \mathbb{Z}$ such that $\sqrt{2} = \frac{a}{b}$.

Since we can always cancel the common factor of a and b in the fractional representation a/b, we can assume that a and b are **coprime**, otherwise just divide both the numerator and denominator by $\gcd(a, b)$ and call the new integers a', b' respectively.

Now $\sqrt{2}b = a$, so $2b^2 = a^2$, this is possible only when a is even, therefore we may set $a = 2a'$ for some $a' \in \mathbb{Z}$, and then

$$2b^2 = 4a'^2 \iff b^2 = 2a'^2,$$

but again this is possible only when b is even. Which means that a, b are both even, and thus cannot be coprime, a contradiction to the first two paragraphs. ∎

Example 4. Let $a \in \mathbb{R}$ be such that the equation $x^3 + \sqrt{2}x^2 - \sqrt{3}x + a = 0$ has three real roots. Prove that the equation has an irrational root.

Sol Suppose that all roots of the polynomial are rational, then we have

$$\begin{aligned} x^3 + \sqrt{2}x^2 - \sqrt{3}x + a &= (x - a_1)(x - a_2)(x - a_3) \\ &= x^3 - (a_1 + a_2 + a_3)x^2 + \cdots. \end{aligned}$$

for some $a_1, a_2, a_3 \in \mathbb{Q}$. By comparing the coefficient, we have $\sqrt{2} = -a_1 - a_2 - a_3 \in \mathbb{Q}$, a contradiction. ∎

Example 5. Let A, B, C, D be sets (you may imagine they are subsets of $\mathbb{R}^2$ for motivation). Prove the following set equalities:

(a) $A \setminus B = A \cap B^c$, where $A, B \subseteq X$ and $B^c = X \setminus B$.

(b) $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$, where $A, B \subseteq X$, $\bullet^c = X \setminus \bullet$.

(c) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

(d) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(e) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(f) $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$, then what is $A \setminus (A \setminus C)$?

(g) $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$

Sol To show two sets A, B are equal, we need to show $A \subseteq B$ and $B \subseteq A$, in other words, we need to prove $x \in A$ iff $x \in B$.

(a) $x \in A \setminus B$
iff ($x \in A$ and $x \notin B$)
iff ($x \in A$ and $x \in B^c$)
iff $x \in A \cap B^c$.

(b) $x \in (A \cup B)^c$
iff $x \notin A \cup B$
iff $\sim(x \in A \cup B)$
iff $\sim$ ($x \in A$ or $x \in B$)
iff ($\sim(x \in A)$ and $\sim(x \in B)$)
iff ($x \notin A$ and $x \notin B$)
iff ($x \in A^c$ and $x \in B^c$)
iff $x \in A^c \cap B^c$.

The other one is essentially the same.

(c) This is essentially the first part in (b). Note that we prefer the notation in this part because the notation $\bullet^c$ being a **relative complement** is **ambiguous!**
Now we have
$x \in A \setminus (B \cup C)$
iff ($x \in A$ and $x \notin (B \cup C)$)
iff $x \in A$ and ($x \notin B$ and $x \notin C$)
iff ($x \in A$ and $x \notin B$) and ($\boldsymbol{x \in A}$ and $x \notin C$)
iff $x \in (A \setminus B) \cap (A \setminus C)$

(d), **(e)** follow from the following fact which can be proved by listing the *truth table* (which we shall not do):

(i) P and (Q or R) = [(P and Q) or (P and R)] (∗)

(ii) P or (Q and R) = [(P or Q) and (P or R)] (∗∗)

Letting $P = x \in A$, $Q = x \in B$ and $R = x \in C$, we get (d) and (e) respectively.

(f) $x \in A \setminus (B \setminus C)$
iff $x \in A$ and $\sim(x \in B \setminus C)$
iff $x \in A$ and $\sim(x \in B$ and $x \notin C)$
iff $x \in A$ and ($x \notin B$ or $x \in C$)
iff ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \in C$) (by (∗))
iff $x \in (A \setminus B) \cup (A \cap C)$. ∎

Example 6. Let $a, b \in \mathbb{Q}$ and $a < b$. Prove that $\exists c \in \mathbb{R} \setminus \mathbb{Q}$ such that $a \le c \le b$.

Sol We prove by contradiction, suppose on the contrary that for every $c \in \mathbb{R} \setminus \mathbb{Q}$, we have $a > c$ or $c > d$, i.e., we have

$$x \notin \mathbb{Q} \implies c < a \text{ or } c > b,$$

which by contrapositive is the same as

$$a \le c \le b \implies x \in \mathbb{Q}.$$

Now consider the number $\sqrt{2}$, which satisfies

$$1 < \sqrt{2} < 2,$$

multiplying both sides by $d = \frac{b-a}{2}$, we have

$$d < \sqrt{2}d < 2d,$$

and therefore

$$a + d < \sqrt{2} + d < a + 2d \implies a < \sqrt{2} + d < b.$$

By our hypothesis which implies $\sqrt{2} + d \in \mathbb{Q}$, but since $a, b \in \mathbb{Q}$, $d = \frac{b-a}{2} \in \mathbb{Q}$, and hence $\sqrt{2} \in \mathbb{Q}$, a contraditcion to Example 3. ∎

Example 7. Let $P(n)$ be a true or false statement. Given $P(1)$ is true, suppose:

$$\forall n \in \mathbb{N}, \text{ if } P(n) \text{ is true, then } P(n+1) \text{ is also true.} \qquad (*)$$

Prove that for each $n \in \mathbb{N}$, $P(n)$ is true.

Sol Suppose on the contrary that $P(m)$ is not true for some $m \in \mathbb{N}$. Let m be the least integer such that $P(m)$ is false. More precisely, we true

$$m = \min\{n \in \mathbb{N} : P(n) \text{ is false}\},$$

the set on the RHS is nonempty by our assumption.

Since $p(1)$ is true, $m \ne 1$, thus $m \ge 2$. As m is the least one, $m - 1 \notin \{n \in \mathbb{N} : P(n) \text{ is false}\}$, i.e., $P(m-1)$ is true. But then by the hypothesis of this example, $P(m)$ must also be true, a contradiction. ∎

Exercises

1. Let A, B, C and D be sets, prove the following set equalities/inequalities:

(a) If $A \subseteq B$, then $A \setminus C \subseteq B \setminus C$.

(b) If $A \subseteq B$, then $C \setminus B \subseteq C \setminus A$.

(c) If $C \ne \emptyset$ and $A \times C = B \times C$, then $A = B$.

(d) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(e) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(f) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$

(g) $A \times B \setminus (C \times D) = [A \times (B \setminus D)] \cup [(A \setminus C) \times (B \cap D)]$

(h) For $a \in \mathbb{R}$ we let $A_a = \{(x, a(x^2 - 1)) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, prove that

$$\bigcap_{a \in \mathbb{R}} A_a = \{(-1, 0), (1, 0)\}.$$

2. Let A and B be sets. If $2^A \subseteq 2^B$, show that $A \subseteq B$.

3. Show that $A \cap B \subseteq C \cup D \implies (A \setminus C) \cap (B \setminus D) = \emptyset$.

4. Is $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \dfrac{2x}{3x^2 + 1}$ injective? Is it surjective?

5. (a) Show that a function $f : A \to B$ is bijective if and only if there is a function $g : B \to A$ such that

$$f \circ g(b) = b, \forall b \in B \quad \text{and} \quad g \circ f(a) = a, \forall a \in A.$$

Such a g is called **the inverse function** of f, denoted by f^{-1}.

(b) Generally how are the graphs of f and f^{-1} related? By considering inverse function, show that

$$\int_1^{\sqrt{2}} \cos^{-1}\left(\frac{1}{x}\right) dx = \frac{\pi}{2\sqrt{2}} - \ln(1 + \sqrt{2})$$

without integration by parts. Here $\cos^{-1}(x)$ is $\arccos(x)$, **NOT** $\dfrac{1}{\cos(x)}$.

6. **(π is irrational)** For the sake of contradiction, let's assume that $\pi = a/b$ for some $a, b \in \mathbb{N}$. For $n = 1, 2, \ldots$ we define $f_n(x) = \frac{1}{n!} x^n (a - bx)^n$ on $\mathbb{R}$.

(a) Show that $f_n(\frac{a}{b} - x) = f_n(x)$ for every x.

(b) Show that for every $k = 0, 1, 2, \ldots$, $f_n^{(k)}(0), f_n^{(k)}(\frac{a}{b}) \in \mathbb{Z}$.

(c) Show that $\int_0^{\pi} f_n(x) \sin x \, dx \in \mathbb{Z}$ for every $n \ge 1$.

(d) Explain why part (c) leads to a contradiction.