Mathematical Induction and Inequalities

We need to know

- who is my TA;
- the standard procedure to go about mathematical induction;
- how to manipulate basic inequalities.

Key definitions and results

Mathematical Induction

Step 1. Show that $P(1)$ is true.

Step 2. Let n be s.t. $P(n)$ is true, then show that $P(n+1)$ is true.

Then we can conclude that $P(n)$ holds for each $n \in \mathbb{N}$.

Variation of Mathematical Induction

Step 1. Show that $P(1)$ is true.

Step 2. Let n be s.t. $P(1), P(2), \dots, P(n)$ are true, then show that $P(n+1)$ is true.

Then we can conclude that $P(n)$ holds for each $n \in \mathbb{N}$.

Remark. We can modify step 1 a little bit. If we prove $P(j)$ is true instead of $P(1)$, then after we finish step 2, we can conclude $P(j), P(j+1), P(j+2), \dots$ are true.

Rules (Inequalities). We need to keep the following in mind:

(a) If $c < d$, we have $c + \boldsymbol{a} < d + \boldsymbol{a}$ and $c - \boldsymbol{a} < d - \boldsymbol{a}$ for any $a \in \mathbb{R}$.

(b) If $p \le q$ and $r < s$, then $p + r < q + s$. [**Caution**: We **cannot** subtract inequalities arbitrarily, e.g., $p - \boldsymbol{r} < q - \boldsymbol{s}$ *can be wrong*.]

(c) If $c < d$, then $\boldsymbol{a}c < \boldsymbol{a}d$ for any $a > 0$
and $\boldsymbol{b}c \boldsymbol{>} \boldsymbol{b}d$ for any $b < 0$.
Most importantly, when $c < d$, $-c > -d$.

(d) If $x > y > 0$, then $0 < \dfrac{1}{x} < \dfrac{1}{y}$. [**Caution**: Given that $a > b > 0$ and $c > d > 0$, we **cannot** divide inequalities, i.e., $\dfrac{a}{c} > \dfrac{b}{d}$ *can be wrong*. However, we **can** multiply, i.e., $ac > bd$ *is true* (note that **positivity is crucial**).]

(e) If $x > y > 0$, then $\sqrt{x} > \sqrt{y} > 0$.
[Actually we have learnt that when $f : A \to B$ is strictly increasing and $x, y \in A$, then whenever $x > y$, $f(x) > f(y)$. The above is the case that $f(x) = \sqrt{x}$.]

Example 1. Prove that for every $n = 1, 2, 3, \dots$, we have

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n}{6}(n+1)(2n+1).$$

<u>Sol</u> Let $P(n)$ denote the statement "$\sum_{k=1}^{n} k^2 = \frac{n}{6}(n+1)(2n+1)$."

Step 1 (Base Case). LHS $= 1^2 = 1$, RHS $= \frac{1}{6}(1+1)(2+1) = 1$, so $P(1)$ holds.

Step 2 (Indutive Step). Assume $P(n)$ holds, i.e., assume that

$$\sum_{k=1}^{n} k^2 = \frac{n}{6}(n+1)(2n+1),$$

we try to show $P(n+1)$ holds. By the way in this step the statmenet $P(n)$ is called **induction hypothesis**.

Indeed,

$$\begin{aligned}\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 \\ &= \frac{n}{6}(n+1)(2n+1) + (n+1)^2 \qquad \text{(by induction hypothesis)} \\ &= (n+1)\left(\frac{n}{6}(2n+1) + n + 1\right) \\ &= \frac{n+1}{6}\big((n+1)+1\big)\big(2(n+1)+1\big),\end{aligned}$$

therefore $P(n+1)$ holds.

By MI, $P(n)$ is true for each $n \in \mathbb{N}$. ∎

Example 2. Let $x_1 = 1$ and $x_{n+1} = \frac{x_n}{2} + \sqrt{x_n}$ for $n = 1, 2, 3, \ldots$. Prove that $0 < x_n < x_{n+1}$ for $n = 1, 2, 3, \ldots$.

Sol **Step 1.** When $n = 1$, we need to show $0 < x_1 < x_2$.

Indeed,

$$x_2 = \frac{x_1}{2} + \sqrt{x_1} = \frac{1}{2} + \sqrt{1} = \frac{3}{2} > 1 = x_1,$$

so $x_2 > x_1 > 0$.

Step 2. Suppose that $0 < x_n < x_{n+1}$, then we have

$$x_{n+2} = \frac{x_{n+1}}{2} + \sqrt{x_{n+1}} > \frac{x_n}{2} + \sqrt{x_n} = x_{n+1} > 0,$$

so by MI we are done. ∎

Example 3. Let $x_1 = 1$ and $x_{n+1} = \frac{2 - x_n}{3 + x_n}$. Prove that for all $k = 1, 2, 3, \ldots$, we have

$$0 < x_{2k} < x_{2k+2} < x_{2k+1} < x_{2k-1}.$$

Sol **Step 1.** Note that

$$x_{n+1} = \frac{2 - x_n}{3 + x_n} = \frac{5}{3 + x_n} - 1,$$

and the first 4 terms are

$$x_1 = 1, \quad x_2 = \frac{1}{4}, \quad x_3 = \frac{7}{13} \quad \text{and} \quad x_4 = \frac{19}{46}.$$

Therefore the statement is true when $k = 1$.

Step 2. Suppose that

$$x_{2k} < x_{2k+2} < x_{2k+1} < x_{2k-1}, \tag{$*$}$$

we need to show $x_{2k+2} < x_{2k+4} < x_{2k+3} < x_{2k+1}$. By $(*)$ we have

$$0 < 3 + x_{2k} < 3 + x_{2k+2} < 3 + x_{2k+1} < 3 + x_{2k-1},$$

now we take reciprocal and mutiply by 5 to get

$$\frac{5}{3 + x_{2k}} > \frac{5}{3 + x_{2k+2}} > \frac{5}{3 + x_{2k+1}} > \frac{5}{3 + x_{2k-1}} > 0,$$

subtracting every term by 1 above, we have

$$x_{2k+1} > x_{2k+3} > x_{2k+2} > x_{2k} > 0.$$

By repeating the above process all over again we have

$$0 < x_{2k+2} < x_{2k+4} < x_{2k+3} < x_{2k+1}.$$

By MI we are done. ∎

> **Example 4 (Finitely Many Propositions).** Suppose an $n\times n$ upper triangular matrix A is invertible, show that A^{-1} is also *upper triangular*.

Sol Let's recall, that A^{-1} is upper triangular is the same as that

$$\boxed{A^{-1}e_k \in \operatorname{span}\{e_1,e_2,\dots,e_k\},\quad \text{for all } k\in\mathbb{N},}$$

here $e_i=(0,\dots,1,\dots,0)^T$, $i=1,2,\dots,n$ denote the standard basis of $\mathbb{R}^n$.

Of course we must require $1\le k\le n$, so **there are just finitely many statements** (or propositions) to be proved. Yet induction is still possible!

Let $A=[a_{ij}]_{n\times n}$ for clarity, now let's finish the standard 2-steps MI procedure.

We will find that naturally the variation of MI is very helpful.

Observation. Since A is upper triangular and invertible, we have

$$\det A = a_{11}a_{22}\cdots a_{nn}\neq 0,$$

therefore $a_{11},a_{22},\dots,a_{nn}\neq 0$.

Step 1 (Base Case). We now show that $A^{-1}e_1\in\operatorname{span}\{e_1\}$.

This case is simple! Since A is upper triangular, we have $Ae_1=a_{11}e_1$, therefore

$$\frac{1}{a_{11}}e_1 = A^{-1}e_1,$$

and hence $A^{-1}e_1\in\operatorname{span}\{e_1\}$, as desired.

Step 2 (Inductive Step).
Unsuccessful MI. Let $k\in\mathbb{N}$ be s.t. $A^{-1}e_k\in\operatorname{span}\{e_1,e_2,\dots,e_k\}$.

Since $A=[a_{ij}]_{n\times n}$, we have

$$\begin{aligned} Ae_{k+1} &= a_{1,k+1}e_1 + a_{2,k+1}e_2+\cdots+a_{k,k+1}e_k + a_{k+1,k+1}e_{k+1}\\ &= \sum_{i=1}^k a_{i,k+1}e_i + a_{k+1,k+1}e_{k+1}.\end{aligned}$$

Now as in the base case we take A^{-1} on both sides to get

$$e_{k+1} = \sum_{i=1}^{k-1} a_{i,k+1}A^{-1}e_i + a_{k,k+1}\boxed{A^{-1}e_k} + a_{k+1,k+1}\boxed{A^{-1}e_{k+1}.} \tag{1}$$

The induction hypothesis provides us the information on $A^{-1}e_k$ but nothing on $A^{-1}e_1,\dots,A^{-1}e_{k-1}$.

We are forced to stop here.

Successful MI. Recall that a variation of MI enables us to **assume more**!

To wit, let's, instead of just one statement, assume k statements:

$$\begin{aligned} &A^{-1}e_1\in\operatorname{span}\{e_1\}\\ &A^{-1}e_2\in\operatorname{span}\{e_1,e_2\}\\ &\quad\cdots\\ &A^{-1}e_k\in\operatorname{span}\{e_1,e_2,\dots,e_k\}.\end{aligned}$$

Now (1) gives us

$$a_{k+1,k+1}A^{-1}e_{k+1} = \underbrace{e_{k+1} - \underbrace{\sum_{i=1}^k a_{i,k+1}\overbrace{A^{-1}e_i}^{\in\operatorname{span}\{e_1,\dots,e_i\}}}_{\in\operatorname{span}\{e_1,\dots,e_k\}}}_{\in\operatorname{span}\{e_1,e_2,\dots,e_{k+1}\}}.$$

Since $a_{k+1,k+1}\neq 0$ (by the observation preceding our MI), so

$$A^{-1}e_{k+1}\in\operatorname{span}\{e_1,\dots,e_{k+1}\},$$

thus by variation of MI, we are done. ∎

Exercises

Exercise 1 to 3 below are assigned by Dr. Li.

1. Prove that for every positive integer n, $1^3+2^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$.

2. Let $x_1=1$ and $x_{n+1}=1-\dfrac{1}{4x_n}$ for all $n=1,2,3,\ldots$. Prove that for all $n=1,2,3,\ldots$, we have $x_n>x_{n+1}>\dfrac{1}{2}$.

3. Let $x_1=5$ and $x_{n+1}=3+\dfrac{4}{x_n}$ for all $n=1,2,3,\ldots$, prove that

$$x_{2k}<x_{2k+2}<x_{2k+1}<x_{2k-1}$$

for all $k=1,2,3,\ldots$.

The following are extra for people who want to try more using M.I..

4. Show that for every $n=1,2,3,\ldots$,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}.$$

Remark. Therefore $1-\frac{1}{2}+\frac{1}{3}-\cdots=\ln 2$ by the definition of Riemann sum!

5. Prove that for every $n=1,2,3,\ldots$ we have

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}\geq\sqrt{n}.$$

Hint: In the inductive step the conclusion is not immediate after the induction hypothesis is used. Try to rewrite the inequality in another *equivalent* form.

6. Prove that for every $n=1,2,3,\ldots$ we have

$$(1+x)^n\geq 1+nx \quad \text{for all } x\geq -1.$$

Remark. This simple convexity-like inequality can be used to show that the sequence $\{(1+\frac{1}{n})^n\}_{n=1}^{\infty}$ is increasing (hence converges!).

7. (Harder) Let $a_1,a_2,\ldots$ be a sequence of real numbers such that $a_{i+j}\leq a_i+a_j$, for all integers $i,j\geq 1$. Prove that for each $n\geq 1$,

$$a_1+\frac{a_2}{2}+\frac{a_3}{3}+\cdots+\frac{a_n}{n}\geq a_n.$$

Hint: Use the variation of Mathematical Induction!