Contents

Chapter 1

A Collection of Problems

Problem marked with **hint** is given a hint at the last section of this problem set. Problem marked with $\otimes$ is the problem that I can't solve, but its technique is worth seeing once.

1.1 Inequality

1.1.1 Cauchy-Schwarz

This part is devoted to the use of Cauchy-Schwarz inequality. Next part (miscellaneous) may require you to use additional inequalities that may or may not be mentioned in preceding questions.

Problem 1. (a) Prove that $(ac+bd)^2 \leq (a^2+b^2)(c^2+d^2)$. When does equality hold?

(b) Hence, or otherwise (except differentiation), compute the maximum value of $f(x) = 7-2x+\frac{1}{2}\sqrt{3+2x-x^2}$, for all $x \in [-1,3]$, when does this mxima occur?

Problem 2. Let a, b, c, d be positive real numbers, prove that

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b} \geq 2.$$

Problem 3. Let $a, b, c > 0$ and $a+b+c = 1$. Prove that

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}} \geq \sqrt{\frac{3}{2}}.$$

Problem 4. Let $a, b, c, d > 0$ and $a+b+c+d = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$, prove that

$$2(a+b+c+d) \geq \sqrt{a^2+3}+\sqrt{b^2+3}+\sqrt{c^2+3}+\sqrt{d^2+3}.$$

Problem 5. Let $a, b, c > 0$ be such that $a^2+b^2+c^2 = 3$, show that

$$\frac{3-a^2}{b+c}+\frac{3-b^2}{c+a}+\frac{3-c^2}{a+b} \geq a+b+c.$$

Problem 6. We would have encountered a famous inequality, the *Nesbitt's inequality* (It appears in Breakthrough algebra p.265, question 2(d)), $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}$, similarly, show that if $abc = 1$,

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \geq \frac{3}{2}.$$

Problem 7. If $a, b, c \le \frac{1}{\sqrt{3}}$ and $a+b+c=1$, show that

$$\sqrt{1-3a^2}+\sqrt{1-3b^2}+\sqrt{1-3c^2} \le \sqrt{6}.$$

Problem 8. If $a, b, c > 0$ and $abc = 1$, show that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(b+a)} \ge \frac{3}{2}.$$

Problem 9. If $a, b, c, d \in \mathbb{R}^+$, $a+b+c+d=3$ and $a^2+2b^2+3c^2+6d^2=5$, find the extreme values of a.

Problem 10. For any positive a, b, c, show that

$$\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2} \ge \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}.$$

Problem 11. If $x, y, z \in \mathbb{R}^+$, and $\frac{x^2}{4}+\frac{(y+1)^2}{9}+\frac{(z+1)^2}{16}=1$, find the extreme values of $2x+y+z-16$.

Problem 12. If $a, b, c \in \mathbb{R}^+$, find the minimum value of $a^2+b^2+(2a-3b-4)^2$.

Problem 13. Let n be positive integer, $\{a_i\}$ be a sequence of positive real number and $S = a_1+a_2+\cdots+a_n$, show that

(a) $\displaystyle\sum_{i=1}^{n}\frac{a_i}{S-a_i} \ge \frac{n}{n-1}$ (b) $\displaystyle\sum_{i=1}^{n}\frac{S-a_i}{a_i} \ge n(n-1).$

Problem 14. If $a_i > 0$, where $i = 1, 2, \ldots, n$, show that

$$\frac{a_1+a_2+\cdots+a_n}{a_1+a_2+\cdots+a_n+1} < \sum_{k=1}^{n}\frac{a_k}{a_k+1}.$$

Problem 15. Suppose $a, b, c > 0$ and $ab+bc+ca = \frac{1}{3}$, show that

$$\frac{a}{a^2-bc+1}+\frac{b}{b^2-ca+1}+\frac{c}{c^2-ab+1} \ge \frac{1}{a+b+c}.$$

Problem 16. Let $x_1, x_2, \ldots, x_n$ be positive real number satisfying $\displaystyle\sum_{k=1}^{n}\frac{1}{x_k} = n$. Find the minimum of $\displaystyle\sum_{k=1}^{n}\frac{(x_k)^k}{k}$.

Problem 17. Let $a, b, c > 0$, prove that

$$\frac{ab}{c(c+a)}+\frac{bc}{a(a+b)}+\frac{ca}{b(b+c)} \ge \frac{a}{c+a}+\frac{b}{a+b}+\frac{c}{b+c}.$$

Problem 18. Let $a, b, c \in (0, 1]$, show

$$\frac{a}{\sqrt{(a^2+b^2)(b^2+c^2)}}+\frac{b}{\sqrt{(b^2+c^2)(c^2+a^2)}}+\frac{c}{\sqrt{(c^2+a^2)(a^2+b^2)}} \ge \frac{3}{2}.$$

Problem 19. Prove that all roots of the polynomial $P(z) = z^n + a_{n-1}z^{n-1}+\cdots+a_1z+a_0$ lie in the open disk

$$\{z : |z| < \sqrt{1+|a_{n-1}|^2+\cdots+|a_1|^2+|a_0|^2}\}.$$

1.1.2 Miscellaneous

Problem 20. Show that $a^{m+n} + b^{m+n} \geq a^m b^n + a^n b^m$, where m and n are non-negative integers.[1]

Problem 21. Given that $0 \leq a, b, c < 1$ and $a + b + c = 2$. Prove that

$$\frac{abc}{(1-a)(1-b)(1-c)} \geq 8.$$

Problem 22. Prove that for any $a, b, c \geq 0$, we always have

$$9(a+b)(b+c)(c+a) \geq 8(a+b+c)(ab+bc+ca).$$

Problem 23. Let $a_i, b_i \geq 0$, show that

$$\sqrt[n]{a_1 a_2 \ldots a_n} + \sqrt[n]{b_1 b_2 \ldots b_n} \leq \sqrt[n]{(a_1+b_1)(a_2+b_2)\ldots(a_n+b_n)}.$$

Problem 24. Show that $\dfrac{1}{a^3+b^3+abc} + \dfrac{1}{b^3+a^3+abc} + \dfrac{1}{b^3+c^3+abc} \leq \dfrac{1}{abc}$.

Problem 25. Given $x, y, z \in \mathbb{R}^+$ and $xy + yz + zx \geq 3$, show that

$$\frac{x^7+y^7}{x^2y+xy^2} + \frac{y^7+z^7}{y^2z+yz^2} + \frac{z^7+x^7}{z^2x+zx^2} \geq 3.$$

Problem 26. If $x, y, z \in \mathbb{R}^+$, show that

$$\frac{x^2}{y^2+z^2+yz} + \frac{y^2}{z^2+x^2+zx} + \frac{z^2}{x^2+y^2+xy} \geq 1.$$

Problem 27. Given $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ and α, β, γ are acute angle, show that

$$\cot^2\alpha + \cot^2\beta + \cot^2\gamma \geq \frac{3}{2}.$$

Problem 28. If x, y, z are positive reals and $x^3 + y^3 + z^3 \leq 3$, show that

$$\frac{1}{(x+y)(x^2+y^2)} + \frac{1}{(y+z)(y^2+z^2)} + \frac{1}{(z+x)(z^2+x^2)} \geq \frac{3}{4}.$$

Problem 29. If $p, q > 0$ and $p^3 + q^3 = 2$, show that $p + q \leq 2$.

Problem 30. If $a, b, c > 0$ and $abc = 1$, show that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Problem 31. Suppose a, b, c are non-negative numbers, if $\dfrac{1}{1+a} + \dfrac{1}{1+b} + \dfrac{1}{1+c} = 1$, show that $abc \geq 8$.

Problem 32. Suppose $a, b > 0$, and $\dfrac{1}{a} + \dfrac{1}{b} = 1$.

(a) Show that $\dfrac{b}{a} + \dfrac{a}{b} \geq 2$.

(b) Hence, or otherwise, show that $a + b \geq 4$.

[1] You may find it helpful in certain questions.

(c) Prove by induction on n, or otherwise, that for all positive integer n,

$$(a+b)^n - a^n - b^n \geq 2^{2n} - 2^{n+1}.$$

Problem 33. Show that if $a, b, c > 0$,

$$\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c}) + (a+b+c)^2 \geq 4\sqrt{3abc(a+b+c)}.$$

Problem 34. Let $x_i \in \mathbb{R}$ and $x_i + x_j \geq x_{i+j}$, for all positive integers i and j, show that

$$x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \cdots + \frac{x_n}{n} \geq x_n, \forall n \in \mathbb{N}.$$

Problem 35. Show that if $\sum_{i=1}^n p_i = 1$ and $p_i, x_i > 0$, then

$$\ln\left(\sum_{i=1}^n p_i x_i\right) \geq \sum_{i=1}^n p_i \ln x_i.$$

Problem 36. Suppose $x \geq y \geq z > 0$ and $a \geq b \geq c > 0$, show that

$$\frac{a^2x^2}{(by+cz)(bz+cy)} + \frac{b^2y^2}{(cz+ax)(cx+az)} + \frac{c^2z^2}{(ax+by)(ay+bx)} \geq \frac{3}{4}.$$

Problem 37. For any real $x_i, y_i \geq 0$, show that

$$\left(\sum_{i=1}^n x_i y_i\right)^3 \leq n\left(\sum_{i=1}^n x_i^3\right)\left(\sum_{i=1}^n y_i^3\right)$$

Problem 38. Given a, b, c are positive, show that if $a+b+c=3$, then

$$\frac{1}{1+2b^2c} + \frac{1}{1+2c^2a} + \frac{1}{1+2a^2b} \geq 1.$$

Problem 39. Let $a, b, c > 0$, show that

$$\frac{\sqrt{a^3+b^3}}{a^2+b^2} + \frac{\sqrt{b^3+c^3}}{b^2+c^2} + \frac{\sqrt{c^3+a^3}}{c^2+a^2} \geq \frac{6(ab+bc+ca)}{(a+b+c)\sqrt{(a+b)(b+c)(c+a)}}.$$

Problem 40. Let $x_1, x_2, \ldots, x_n \in \mathbb{R}^+$, prove that

$$\frac{x_1x_2\cdots x_n}{(x_1+x_2+\cdots+x_n)^n} \leq \frac{(1+x_1)\cdots(1+x_n)}{(n+x_1+x_2+\cdots+x_n)^n}.$$

Problem 41. Let $a, b, c \geq 0$ and $t \in (0,3]$. Prove that

$$(3-t) + t(abc)^{2/t} + a^2 + b^2 + c^2 \geq 2(ab+bc+ca).$$

Problem 42. For any positive a_i, b_i, $i = 1, 2, \ldots, n$, show that

$$\left(\sum_{i=1}^n a_i b_i^2\right)^3 \leq \left(\sum_{i=1}^n a_i^3\right)\left(\sum_{i=1}^n b_i^3\right)^2.$$

Problem 43. Let $x, y, z > 1$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$. Show that

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

Problem 44. Let x, y, z be three non-negative numbers satisfying $xyz = 1$, show that

$$\sqrt{4+9x^2} + \sqrt{4+9y^2} + \sqrt{4+9z^2} \le \sqrt{13}(x+y+z).$$

Problem 45. For any positive x, y, z, prove that

$$\frac{xy}{z(z+x)} + \frac{yz}{x(x+y)} + \frac{zx}{y(y+z)} \ge \frac{x}{z+x} + \frac{y}{x+y} + \frac{z}{y+z}.$$

Problem 46. Let a, b, c be positive real numbers satisfying the condition $a+b+c = 3$, prove that

$$\frac{a^2(b+1)}{a+b+ab} + \frac{b^2(c+1)}{b+c+bc} + \frac{c^2(a+1)}{c+a+ca} \ge 2.$$

Problem 47. Let $a, b, c \in \mathbb{R}^+$ be such that $a+b+c > 2$. Prove that

$$\sqrt{\frac{a^2+b^2}{2c+a+b-2}} + \sqrt{\frac{b^2+c^2}{2a+b+c-2}} + \sqrt{\frac{c^2+a^2}{2b+c+a-2}} \ge 3.$$

Problem 48. If $a, b, c > 1$, show that

$$\frac{a(a^3-1)}{a-1} + \frac{b(b^3-1)}{b-1} + \frac{c(c^3-1)}{c-1} \le \frac{abc-1}{\sqrt[3]{abc}-1}\left(\frac{a^4+b^4+c^4}{abc}\right).$$

Problem 49. (Generalization of Cauchy-Schwarz inequality) Let $a_{i_j} > 0$, $i, j = 1, 2, \ldots, n$. Show that

$$\left(\sum_{i=1}^{n} a_{1_i}^m\right)\left(\sum_{i=1}^{n} a_{2_i}^m\right)\cdots\left(\sum_{i=1}^{n} a_{m_i}^m\right) \ge \left(\sum_{i=1}^{n} a_{1_i}a_{2_i}\cdots a_{m_i}\right)^m$$

Problem 50. Let $a, b, x, y > 0$ be such that $1 \ge a^{11} + b^{11}$ and $1 \ge x^{11} + y^{11}$, show that $1 \ge a^5x^6 + b^5y^6$.

Problem 51. (Kyiv 2006) Let $x, y, z > 0$ be such that $xy + yz + zx = 1$. Prove that

$$\frac{x^3}{1+9y^2xz} + \frac{y^3}{1+9z^2yx} + \frac{z^3}{1+9x^2zy} \ge \frac{(x+y+z)^3}{18}.$$

Problem 52. Let a, b, c, m, n be positive real numbers. Prove that

$$\frac{a^2}{b(ma+nb)} + \frac{b^2}{c(mb+nc)} + \frac{c^2}{a(mc+na)} \ge \frac{3}{m+n}.$$

Problem 53. Let $a, b, c > 0$ be the sidelengths of a triangle. Prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \ge 0.$$

Problem 54. Let $a, b, c > 0$, show that

$$(a^3+1)(b^3+1)(c^3+1) \ge (a^2b+1)(b^2c+1)(c^2a+1).$$

Problem 55. If $a, b, c > 0$, prove that

$$\frac{a+\sqrt{ab}+\sqrt[3]{abc}}{3} \le \sqrt[3]{a\cdot\frac{a+b}{2}\cdot\frac{a+b+c}{3}}.$$

Problem 56. Let $a, b, c > 0$, prove that

$$\frac{a^6}{b^2+c^2} + \frac{b^6}{c^2+a^2} + \frac{c^6}{a^2+b^2} \ge \frac{abc(a+b+c)}{2}.$$

Problem 57. Let f be convex on $[a, b]$. If $c, d \in [a, b]$ with $c - a > b - d$, prove that

$$2f\left(\frac{c+d}{2}\right) \le f(c) + f(d) \le f(c+d-b) + f(b).$$

Problem 58. Let A, B, C be angles of a triangle. Prove that

$$\frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C} \le \frac{9\sqrt{3}}{2\pi}.$$

Problem 59. Let x, y, z be non-negative integers with $x + y + z = 1$, prove that

$$0 \le xy + yz + zx - 2xyz \le \frac{7}{27}.$$

Problem 60. Prove that for any $a, b, c \in \mathbb{R}$,

$$\sqrt{a^2 + (1-b)^2} + \sqrt{b^2 + (1-c)^2} + \sqrt{c^2 + (1-a)^2} \ge \frac{3\sqrt{2}}{2}.$$

Problem 61. (**Mircea Lascu**) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{b+c}{\sqrt{a}} + \frac{c+a}{\sqrt{b}} + \frac{a+b}{\sqrt{2}} \ge \sqrt{a} + \sqrt{b} + \sqrt{c} + 3.$$

Problem 62. Let a, b, c, x, y, z be positive real numbers such that $x + y + z = 1$. Prove that

$$ax + by + cz + 2\sqrt{(xy + yz + zx)(ab + bc + ca)} \le a + b + c.$$

Problem 63. Let $x, y, z > 0$ and $x + y + z = 1$, show that

$$\frac{x}{xy+1} + \frac{y}{yz+1} + \frac{z}{zx+1} \ge \frac{36xyz}{13xyz+1}.$$

Problem 64. Let $x_1, x_2, \ldots, x_n > 0$, $m, n \in \mathbb{N}$, $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{Q}^+$ and $\beta = \beta_1 + \beta_2 + \cdots + \beta_n$.

(a) $\dfrac{\sum_{i=1}^n x_i}{n} \le \left(\dfrac{\sum_{i=1}^n x_i^2}{n}\right)^{\frac{1}{2}} \le \left(\dfrac{\sum_{i=1}^n x_i^3}{n}\right)^{\frac{1}{3}}$

(b) $\dfrac{\sum_{i=1}^n x_i}{n} \le \left(\dfrac{\sum_{i=1}^n x_i^m}{n}\right)^{1/m}$

(c) $\dfrac{\sum_{i=1}^n \beta_i x_i}{\beta} \le \left(\dfrac{\sum_{i=1}^n \beta_i x_i^m}{\beta}\right)^{1/m}$

One solution uses Jensen's inequality, that certainly kills the problem instantly. You can suppose yourself are merely aware of Cauchy-Schwarz inequality, as a challenge.

Problem 65. Let $a, b, c > 0$ and $abc = 1$. Prove that

$$\frac{a}{a+b+1} + \frac{b}{b+c+1} + \frac{c}{c+a+1} \ge 1.$$

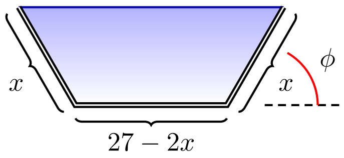
Problem 66. Let $x, y, z > 0$ and $x + y + z = 1$. Prove that

$$\sqrt{\frac{x}{yz}} + \sqrt{\frac{y}{zx}} + \sqrt{\frac{z}{xy}} \ge 2\left(\sqrt{\frac{x}{(x+y)(x+z)}} + \sqrt{\frac{y}{(y+z)(y+x)}} + \sqrt{\frac{z}{(z+x)(z+y)}}\right).$$

Problem 67. When $a+b+c=3$, $a,b,c \geq 0$, prove that

$$\frac{a+3}{3a+bc}+\frac{b+3}{3b+ca}+\frac{c+3}{3c+ab} \geq 3.$$

Problem 68. A length of sheet metal 27 inches wide is to be made into a water trough by bending up two sides as shown in the accompanying figure. Find x and ϕ so that the trapezoid-shaped cross section has a maximum area.



Restriction: You cannot use differentiation, any elementary approach is fine.

Problem 69. Show that for any $n \geq 2$, if $a_1, a_2, \ldots, a_n > 0$, then

$$(a_1^3+1)(a_2^3+1)\cdots(a_n^3+1) \geq (a_1^2a_2+1)(a_2^2a_3+1)\cdots(a_n^2a_1+1).$$

Problem 70. (**Austrian Mathematical Olympiad 2008**) Prove that the inequality $\sqrt{a^{1-a}b^{1-b}c^{1-c}} \leq \frac{1}{3}$ holds for all positive real numbers a,b,c with $a+b+c=1$.

Moreover, try to generalize this inequality and argue that for any $a_i > 0$,

$$\frac{a_1+a_2+\cdots+a_n}{n} \geq \left(\frac{a_1a_2\cdots a_n}{(a_1^{a_1}a_2^{a_2}\cdots a_n^{a_n})^{1/\sum_{i=1}^n a_i}}\right)^{1/(n-1)}.$$

Remark. *This inequality is useful if we are given a condition on* $\prod a_i^{a_i}$ *(especially* $=1$*).*

Problem 71. Let $a,b,c>0$, $\left(\frac{1}{a^2}+1\right)\left(\frac{1}{b^2}+1\right)\left(\frac{1}{c^2}+1\right)=512$ and $k=a+b+c$, find the minimum value of k.

Problem 72. Given $a,b,c>0$. Prove that

$$\frac{a^3+b^3+c^3}{2abc} \geq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

Problem 73. Let x,y,z be real numbers greater than or equal to 1. Prove that

$$(x^2-2x+2)(y^2-2y+2)(z^2-2z+2) \leq (xyz)^2-2xyz+2.$$

Problem 74. Let $a_1,a_2,\ldots,a_n$ and $b_1,b_2,\ldots,b_n$ be positive reals and $a=\sum_{i=1}^n a_i, b=\sum_{i=1}^n b_i$. Prove that:

$$\sum_{i=1}^n \frac{a_ib_i}{a_i+b_i} \leq \frac{ab}{a+b}.$$

Problem 75. Show that if $a,b,c \geq 0$, then

$$(a+b+c)(a^2+b^2+c^2)+9abc \geq 2(a+b+c)(ab+bc+ca).$$

Problem 76. Let $a,b,c \geq 0$, deduce that

$$a^2+b^2+c^2+2abc+1 \geq 2(ab+bc+ac).$$

Problem 77. Let $a,b,c \geq 0$, prove that

$$a^2+b^2+c^2+2abc+3 \geq (1+a)(1+b)(1+c).$$

Problem 78. Let $x, y, z > 0$, prove that

$$\frac{xy}{x^2+y^2+2z^2}+\frac{yz}{y^2+z^2+2x^2}+\frac{zx}{z^2+x^2+2y^2}\le\frac{3}{4}.$$

Problem 79. Let $a, b, c > 0$ and $abc = 1$. Prove that

$$\frac{1}{a+b^2+c^3}+\frac{1}{b+c^2+a^3}+\frac{1}{c+a^2+b^3}\le 1.$$

Problem 80. Let $a_i \in [a, A]$ and $b_i \in [b, B]$ with $a, b > 0$, prove that

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \le \frac{1}{4}\left(\sqrt{\frac{AB}{ab}}+\sqrt{\frac{ab}{AB}}\right)\left(\sum_{i=1}^{n} a_i b_i\right)^2.$$

Problem 81. Let $\{a_1, a_2, \ldots, a_n\} = \{1, 2, \ldots, n\}$. Prove that

$$\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n-1}{n}\le\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_{n-1}}{a_n}.$$

Problem 82. Let a, b, c be positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$. Prove

$$a^2+b^2+c^2\ge 2a+2b+2c+9.$$

Problem 83. Let a, b, c be non-negative numbers with $a + b + c = 3$. Prove that

$$\frac{a}{b^2+1}+\frac{b}{c^2+1}+\frac{c}{a^2+1}\ge\frac{3}{2}.$$

Problem 84. Let $x, y, z > 0$ and $xyz + xy + yz + zx + 2$, prove that

$$\sqrt{x}+\sqrt{y}+\sqrt{z}\le\frac{3}{2}\sqrt{xyz}.$$

1.2 Integration

Problem 85. Numerical answer is not allowed.

(a) $5050\cdot\dfrac{\int_0^1 (1-x^{50})^{100}\,dx}{\int_0^1 (1-x^{50})^{101}\,dx}$

(b) $\displaystyle\int_0^1 \frac{(1-2x)e^x+(1+2x)e^{-x}}{(e^x+e^{-x})^3}\,dx$

(c) $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\sin nx}{(2^x+1)\sin x}\,dx, \forall n\in\mathbb{Z}$

(d) $\displaystyle\int_{-\pi}^{\pi} \frac{\sin^2 nx}{(e^x+1)\sin^2 x}\,dx, \forall n\in\mathbb{N}$

(e) $\displaystyle\int_0^{\infty} \frac{\tan^{-1}(\pi x)-\tan^{-1} x}{x}\,dx$

(f) $\displaystyle\int_0^{\infty} \frac{\cos x-1}{xe^x}\,dx$

(g) $\displaystyle\int_0^1 \frac{\tan^{-1} x}{x\sqrt{1-x^2}}\,dx$

(h) $\displaystyle\int_0^1 \frac{x^b-x^a}{\ln x}\,dx$, here $a\in(0,b)$

(i) $\displaystyle\int_0^1 \sin\left(\ln\frac{1}{x}\right)\cdot\frac{x^b-x^a}{\ln x}\,dx$

(j) Find $\displaystyle\int \frac{x^2-1}{(x^2+1)\sqrt{1+x^4}}\,dx.$

(k) By using Poisson integral formula, find

$$\int_0^{2\pi}\frac{1}{5-4\cos t}\,dt\ \&\ \int_0^{2\pi}\frac{3e^{\cos t}\cos(\sin t)}{5-4\cos(t-\sqrt{2})}\,dt.$$

(l) $\displaystyle\int_0^{\pi} \frac{\cos nx-\cos na}{\cos x-\cos a}\,dx,\ a\in(0,\pi), n\in\mathbb{N}$

Problem 86. Let f be continuous. We say that $f \in L^p(\mathbb{R})$ $(1 \le p \le \infty)$ if

$$\|f\|_p := \begin{cases} \left(\int_{\mathbb{R}} |f(x)|^p\,dx\right)^{1/p} < \infty, & 1 \le p < \infty, \\ \sup_{x\in\mathbb{R}} |f(x)| < \infty, & p = \infty \end{cases}$$

is finite.

(a) (**Young inequality**) Let two functions $f, g \in L^p(\mathbb{R})$ be continuous with $1 \le p \le \infty$. Prove that

$$\|f * g\|_p \le \|f\|_1 \|g\|_p,$$

where the function $f * g$ is the convolution of f and g defined by $f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dy$.

(b) (**Sobolev inequality**) Let $f(x)$ be continuously differentiable (i.e., it has continuous derivative) on $\mathbb{R}$. Assume $f, f' \in L^2(\mathbb{R})$ and $\lim_{|x|\to\infty} f(x) = 0$. Prove that

$$\|f\|_\infty \le \sqrt{2}\|f\|_2^{1/2}\|f'\|_2^{1/2}.$$

Problem 87. Suppose $f(x)$ is an integrable function on $[a, b]$, prove that the following are equivalent.

(a) $\int_c^d f(x)\,dx = 0$ for any $[c, d] \subset [a, b]$.

(b) $\int_a^b |f(x)|\,dx = 0$.

(c) $\int_a^b f(x)g(x)\,dx = 0$ for any continuous function $g(x)$.

(d) $\int_a^b f(x)g(x)\,dx = 0$ for any integrable function $g(x)$.

(e) $f(x) = 0$ at continuous points.

Problem 88. If the function $f(x)$ satisfies $f(0) = 5$ and $f'(x) = 6x + \sqrt{2 + x^2}\sin^2 x$, find $\int_{-2}^{2} f(x)\,dx$.

Problem 89. Show that $\displaystyle\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{n(n+1)\cdots(n+k)} = \int_0^1 \frac{e^x - 1}{x}\,dx$.

Problem 90. Suppose $f(x)$ is continuous on $[a, b]$, differentiable on (a, b), $f(a) = 0$ and $0 \le f'(x) \le 1$. Prove that

$$\left(\int_a^b f(x)\,dx\right)^2 \ge \int_a^b f(x)^3\,dx.$$

Problem 91. Let $f(x)$ be a integrable function continuous at 0, show that $\displaystyle\lim_{h\to 0^+}\int_0^1 \frac{h}{h^2 + x^2} f(x)\,dx = \frac{\pi}{2} f(0)$.

Problem 92. Suppose f is an non-negative continuous function on $[a, b]$, show that $\lim_{n\to\infty} \left(\int_a^b f(x)^n \, dx \right)^{\frac{1}{n}} = \max_{a \le x \le b} f(x)$.

Problem 93. Show that $\int_{-1}^{1} (1 - x^2)^n \, dx \ge \frac{4}{3\sqrt{n}}$, $\forall n \in \mathbb{N}$.

Problem 94. Let $[x]$ be the biggest integer $\le x$. Let $a > 0$. Determine the convergence of the improper integral

$$\int_0^1 \left(\left[\frac{a}{x} \right] - a \left[\frac{1}{x} \right] \right) dx.$$

Problem 95. (A well-known integral without University knowledge)

(a) Prove that

$$\begin{cases} 1 - x^2 \le e^{-x^2} & \text{when } 0 \le x \le 1 \\ e^{-x^2} \le \dfrac{1}{1+x^2} & \text{when } x \ge 0. \end{cases}$$

(b) Hence show that $\lim_{\lambda\to\infty} \int_0^{\lambda} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$.

Problem 96. Let $f(x)$ be a differentiable function, $f(0) = 0$ and $f(1) = 1$, prove that $\int_0^1 |f(x) - f'(x)| \, dx \ge \frac{1}{e}$.

Problem 97. Let $f(x)$ be differentiable function, $f(1) = 1$ and $f'(x) = \frac{1}{x^2 + f(x)^2}$, for all $x \ge 1$. Show that $\lim_{x\to\infty} f(x)$ exists and $\lim_{x\to\infty} f(x) \le 1 + \frac{\pi}{4}$.

Problem 98. Let $f(x)$ be continuous and increasing on $[a, b]$, prove that $\int_a^b x f(x) \, dx \ge \left(\frac{a+b}{2} \right) \int_a^b f(x) \, dx$.

Problem 99. Suppose $f(x)$ and $g(x)$ are integrable on $[a, b]$. Prove that for any $\epsilon > 0$, there is $\delta > 0$, such that for any partition P satisfying $\|P\| < \delta$ and choices $x_i^*, x_i^{**} \in [x_{i-1}, x_i]$, we have

$$\left| \sum f(x_i^*) g(x_i^{**}) \Delta x_i - \int_a^b f(x) g(x) \, dx \right| < \epsilon.$$

Problem 100. Let f be integrable on $[0, 1]$. Suppose there is positive real numbers m and M such that $m \le f(x) \le M$, for all $x \in [0, 1]$, then prove that

$$\int_0^1 f(x) \, dx \int_0^1 \frac{1}{f(x)} \, dx \le \frac{(m+M)^2}{4mM}.$$

Problem 101. Let $x(t)$ be continuous on $[0, a]$ satisfying

$$|x(t)| \le M + k \int_0^t |x(u)| \, du,$$

where M and k are positive constants, prove that $|x(t)| \le M e^{kt}$, for $t \in [0, a]$.

Problem 102. Let f be a continuous function on $[-1,1]$. Suppose for any even function g on $[-1,1]$, the integral

$$\int_{-1}^{1} f(x)g(x)\,dx = 0,$$

then prove that f is an odd function on $[-1,1]$.

Problem 103. Let f be continuous on $[0,\pi]$, $n \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty} \int_0^{\pi} f(x)|\sin nx|\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx.$$

Problem 104. Let f be continuous real-valued function on $[0,1]$ and $f(0)=0,\ f(1)=1$. Find

$$\lim_{n\to\infty} n\int_0^1 f(x)x^{2n}\,dx.$$

Problem 105. Find $\lim\limits_{n\to\infty} n^2\left(\int_0^1 \sqrt[n]{1+x^n}\,dx - 1\right)$.

Problem 106. Find $\lim\limits_{n\to\infty} \frac{1}{n^4}\left(\sum\limits_{k=1}^{n} k^2\int_k^{k+1} x\ln\big((x-k)(k+1-x)\big)\,dx\right)$.

Problem 107. Let $f:[0,1]\to\mathbb{R}$ be differentiable on $[0,1]$ with $f(1)=0$, show that

$$\lim_{n\to+\infty} n^2\int_0^1 f(x)x^n\,dx = -f'(1).$$

1.3 Evaluation of Limit

Problem 108. Evaluate the following limits

(a) $\lim\limits_{x\to 0} \frac{\sqrt{\cos 2x}\,\sqrt[3]{\cos 3x}\cdots\sqrt[n]{\cos nx}-1}{x^2}$

(b) $\lim\limits_{x\to 0} \frac{\sqrt{1-2x}\,\sqrt[3]{1-3x}\cdots\sqrt[n]{1-nx}}{\sin x}$

(c) $\lim\limits_{x\to 0} \frac{(1+2x+x^2)^{\frac{1}{x}}-(1+2x-x^2)^{\frac{1}{x}}}{x}$

(d) $\lim\limits_{\substack{n\to\infty\\ \alpha>1}} n\left(\frac{1^\alpha+2^\alpha+...+n^\alpha}{n^{\alpha+1}}-\frac{1}{\alpha+1}\right)$,

Problem 109. Let $a_1, a_2, \dots$ be a sequence of positive real numbers. Prove that

$$\lim_{n\to\infty} \frac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)} = 0.$$

Problem 110. Let $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} b_n = b$, prove that $\lim\limits_{n\to\infty} \frac{a_1b_n + a_2b_{n-1}+\cdots+a_nb_1}{n} = ab$.

Problem 111. Let $x_1 = \sin x_0 > 0$, $x_{n+1} = \sin x_n$, $n > 1$, prove that $\lim\limits_{n\to\infty} \sqrt{\frac{n}{3}}x_n = 1$.

Problem 112. Let $a_1 = 1$ and $a_{n+1} = \sqrt{a_1+a_2+a_3+.....+a_n}$, for $n>0$, find $\lim\limits_{n\to\infty} \frac{a_n}{n}$.

Problem 113. Let $n,k\in\mathbb{N}$, $t_{nk}\ge 0$, $\sum\limits_{k=1}^{n} t_{nk} = 1$ and $\lim\limits_{n\to\infty} t_{nk} = 0$. Suppose $\lim\limits_{n\to\infty} a_n = a$, let $x_n = \sum\limits_{k=1}^{n} t_{nk}a_k$, prove that $\lim\limits_{n\to\infty} x_n = a$.

Problem 114. Let $\lim_{n\to\infty} a_n = a$, prove that $\lim_{n\to\infty} \frac{a_1 + 2a_2 + \cdots + na_n}{n^2} = \frac{a}{2}$.

Problem 115. Suppose $f(x)$ has second order derivative at 0. Let $\lim_{x\to 0}\left(1 + x + \frac{f(x)}{x}\right)^{\frac{1}{x}} = e^\lambda$.

(a) Find $f(0)$, $f'(0)$ and $f''(0)$.

(b) Find $\lim_{x\to 0}\left(1 + \frac{f(x)}{x}\right)^{\frac{1}{x}}$.

Problem 116. Discuss the continuity of the function

$$f(x,y) = \begin{cases} \dfrac{|x|^p|y|^q}{(|x|^k + |y|^l)^\alpha(|x|^m + |y|^n)^\beta} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

at $(0,0)$. All parameters are positive.

Problem 117. Let $p, q > 0$. Study the continuity and differentiability of the function

$$f(x,y) = \begin{cases} \dfrac{|x|^p|y|^q}{|x| + |y|} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

at all the points on the plane.

1.4 Sequence and Series

Problem 118. Denote $[x]$ the greatest integer not exceeding x.

(a) Calculate $\sum_{k=1}^{100} [\sqrt{k(k+4)+20}]$.

(b) Express $\sum_{k=1}^{n} [\sqrt{k}]$ in terms of n and $a = [\sqrt{n}]$.

Problem 119. Given that $u = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots$, $v = \frac{x}{1!} + \frac{x^4}{4!} + \frac{x^7}{7!} + \cdots$, $w = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots$, show that $u^3 + v^3 + w^3 - 3uvw = 1$. Moreover, find the function that u converges to.

Problem 120. Show that $\sum_{k=0}^{n-1} (-1)^k \cos^n\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}$, for any positive integer n.

Problem 121. For any $x_i > -1$, $i \in \mathbb{N}$.

(a) Prove that if $\sum_{n=1}^{\infty} x_n$ converges, then $\prod_{n=1}^{\infty}(1 + x_n)$ converges if and only if $\sum_{n=1}^{\infty} x_n^2$ converges.

(b) Prove that if $\sum_{n=1}^{\infty} x_n^2$ converges, then $\prod_{n=1}^{\infty}(1 + x_n)$ converges if and only if $\sum_{n=1}^{\infty} x_n$ converges.

Problem 122. Evaluate $f(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} \tan\frac{x}{2^k}$ (i.e. find f), where f is well-defined on its own domain.

Problem 123. Show that if $\sum_{n=1}^{\infty} a_n$ convereges, then we have $\lim_{n\to\infty} \dfrac{a_1 + 2a_2 + \cdots + na_n}{n} = 0$.

Problem 124. (Generalized version of the above question) Suppose $\{b_n\}$ is increasing with $\lim_{n\to\infty} b_n = \infty$ and $\sum_{n=1}^{\infty} a_n$ converges, show that $\lim_{n\to\infty} \dfrac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{b_n} = 0$.

Problem 125. Suppose $\sum_{n=1}^{\infty} a_n$ converges, let $b_n = \dfrac{a_1 + 2a_2 + \cdots + na_n}{n(n+1)}$, show that $\sum_{n=1}^{\infty} b_n$ also converges. Moreover, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$.

Problem 126. Does the following converge?

(a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \sqrt{\ln \frac{n+1}{n}} \right)$ (b) $\displaystyle\sum_{n=1}^{\infty} (n^{1/(n^2+1)} - 1)$

Problem 127. [2010 IMO prelim (held in 2009)] Evaluate $\tan \left(\sum_{n=1}^{2009} \tan^{-1} \dfrac{1}{2n^2} \right)$.

Problem 128. Evaluate $\sum_{n=0}^{\infty} \cot^{-1}(n^2 + n + 1)$.

Problem 129. Let $a_1 = 1$, $a_n = n(a_{n-1} + 1)$, $n = 2, 3, \ldots$, prove that $\prod_{n=1}^{\infty} \left(1 + \dfrac{1}{a_n}\right) = e$.

Problem 130. Find the function $T(x)$ satisfying $T(x) = T\left(\dfrac{x}{2}\right) + b \cdot x \log x$, $T(1) = 1$ and $\lim_{x\to 0^+} T(x)$ exists.

Problem 131. Find all $p \in \mathbb{R}$ such that $\sum_{k=2}^{\infty} \dfrac{1}{(\log \log k)^{p \log k}}$ converges.

Problem 132. Let $0 < x_1 < 1$ and define $x_{n+1} = x_n(1 - x_n)$. Show that the series $\sum x_n$ diverges.

hint **Problem 133.** Let $\{a_n\}$ be a sequence of non-negative real numbers with the property that for every sequence $\{b_n \geq 0\}$ with $\sum_{n=1}^{\infty} b_n^2 < \infty$ one has $\sum_{n=1}^{\infty} a_n b_n < \infty$. Prove that $\sum_{n=1}^{\infty} |a_n|^2 < \infty$.

⊗**Problem 134.** Let $a_1, a_2, \ldots, a_M \in \mathbb{C}$, prove that

$$\overline{\lim_{n\to\infty}} \left| \sum_{j=1}^{n} a_j^n \right|^{1/n} = \max_{j=1,2,\ldots,M} |a_j|.$$

Problem 135. Let $a_1, a_2, a_3, \ldots$ be a decreasing sequence of positive real numbers. Let $s_n = a_1 + a_2 + \cdots + a_n$ and $b_n = \dfrac{1}{a_{n+1}} - \dfrac{1}{a_n}$, $n \geq 1$. Prove that if the sequence $\{s_n\}$ is convergent then the sequence $\{b_n\}$ is unbounded.

Problem 136. Consider the sequence $\{a_n\}_{n\geq 1}$ such that $a_1 = a_2 = 0$ and $a_{n+1} = \frac{1}{3}(a_n + a_{n-1}^2 + b)$, where $0 \leq b \leq 1$. Prove that the sequence is convergent and evaluate $\lim_{n\to\infty} a_n$.

Problem 137. Consider a sequence of positive real numbers $\{a_n\}_{n=1}^{\infty}$ such that $a_{n+1} - \dfrac{1}{a_{n+1}} = a_n + \dfrac{1}{a_n}$, for all $n \geq 1$. Compute

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \right).$$

Problem 138. Let $\{x_n\}$ be a bounded sequence of real numbers such that

$$\lim_{n\to\infty}(x_{n+1} - x_n) = 0, \quad \underline{\lim}_{n\to\infty} x_n = a \quad \text{and} \quad \overline{\lim}_{n\to\infty} x_n = b.$$

Show that for every $c \in [a, b]$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $\lim_{i\to\infty} x_{n_i} = c$.

Problem 139. Let $a_0 \neq 0$ and $\{a_n\}$ be a sequence of real numbers defined by

$$a_{n+2} = \frac{n(n+1) - \ell(\ell+1)}{(n+1)(n+2)} a_n, \quad n \geq 0, \ell \in \mathbb{R} \setminus \{0, 2, 4, 6, \ldots\}.$$

Prove that the series $a_0 + a_2 + a_4 + \cdots$ diverges.

1.5 Binomial Identity

Problem 140. Show that $\displaystyle\sum_{k=0}^{2009} \binom{2009}{k} \frac{(-1)^k}{k+2011} = \sum_{k=0}^{2010} \binom{2010}{k} \frac{(-1)^k}{k+2010}$.

Problem 141. Show that when $|4x| < 1$, $(1-4x)^{-1/2} = \displaystyle\sum_{n=0}^{\infty} \binom{2n}{n} x^n$.

Problem 142. Show that $\displaystyle\sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} = 4^n$.

Problem 143. Show that $\displaystyle\sum_{k=0}^{n} \frac{1}{1-2k} \binom{2k}{k} 2^{2n-2k} = \binom{2n}{n}$.

hint **Problem 144.** Prove that If $a_n = \displaystyle\sum_{k=1}^{n} (-1)^k \binom{n}{k} b_k$, then $b_k = \displaystyle\sum_{l=1}^{k} (-1)^l \binom{k}{l} a_l$, where $k = 1, 2, \ldots, n$.

Problem 145. Prove that $\displaystyle\sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k} \binom{n}{k} = 1 + 2 + \cdots + \frac{1}{n}$. Hence, or otherwise, show that

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right) = -\frac{1}{n}.$$

Problem 146. (i) Give a combinatorial interpretation to the equality $\dbinom{n}{k}\dbinom{k}{m} = \dbinom{n}{m}\dbinom{n-m}{k-m}$.

(ii) Let $m \leq n$ and $\delta_{m,n} = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$ Prove that $\displaystyle\sum_{k=m}^{n} (-1)^k \binom{n}{k} \binom{k}{m} = (-1)^m \delta_{m,n}$.

(iii) Prove by using (i) and (ii) that $\displaystyle\sum_{k=m}^{n} (-1)^k \frac{1}{k+1} \binom{n}{k} \binom{k}{m} = \frac{(-1)^m}{n+1}$.

hint **Problem 147.** Show that $\sum_{i=1}^{m-1}\sum_{j=1}^{m-i}\binom{m}{i}\binom{m-i}{j}(n-2)^{m-i-j} = n^m - 2(n-1)^m + (n-2)^m$, likewise,

$$\sum_{i=1}^{m-2}\sum_{j=1}^{m-i}\sum_{r=1}^{m-i-j}\binom{m}{i}\binom{m-i}{j}\binom{m-i-j}{r}(n-3)^{m-i-j-r} = n^m - 3(n-1)^m + 3(n-2)^m - (n-3)^m.$$

1.6 Basic Counting

Problem 148. In how many ways can we choose elements from $\{1, 2, 3, \ldots, n\}$ to form 2 disjoint subsets?

Problem 149. There are 6 distinct presents. In how many ways can we distribute the presents to 3 people if everyone at least have one present?

Problem 150. There are 20 distinct presents. In how many ways can we distribute the presents to 6 people if everyone at least have one present? You must give your numerical answer in exact!

Problem 151. Suppose that a mathematical expression can only be formed by the following symbols: $0, 1, 2, \ldots, 9, \times, +, \div$. Some examples are "$0+9$"; "$2+28$"; "$100 \div 5 + 6$". Let a_n be the the number of such mathematical expression of length n (e.g. "$0+9$" is considered of length 3). Find a recurrence relation for a_n and compute the closed form for a_n.

Problem 152. Each of two fair dices is tossed 6 times, let $a_1, a_2, \ldots, a_6$ and $b_1, b_2, \ldots, b_6$ be the values shown in the first and second dice respectively. Find the probability that $\sum_{i=1}^{6} a_i \neq \sum_{i=1}^{6} b_i$.

Problem 153. Let there be 10 people whose ages are ranged from 1 to 60 (with 1 and 60 included). Suppose their ages are pairwise distinct, show that there is a possible way to divide these people into 2 groups such that, the sums of ages of each group are the same.

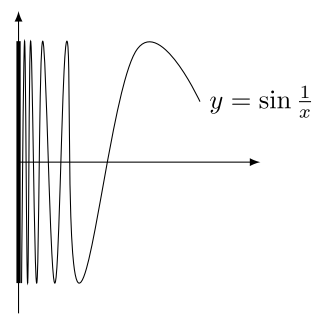
For example, if there are 1, 2, 3 year-old people out of 10 people, then the division $\{3\}$ and $\{1, 2\}$ are allowed since $3 = 1 + 2$ (no need to group all of 10 people).

Problem 154. If three tickets are chosen at random without replacement from a set of $6n$ tickets numbered respectively $1, 2, \ldots, 6n$, what is the probability that the sum of the numbers on the numbers on the chosen tickets is $6n$?

1.7 Function and Differentiation

1.7.1 Real-valued Function

Problem 155. Let $G = \{(x, \sin\frac{1}{x}) : x \in (0, 1]\}$ and let $S = \{0\} \times [-1, 1]$. Define $M = G \cup S$. Show that there cannot be any continuous path in M connecting point in G and point in S.



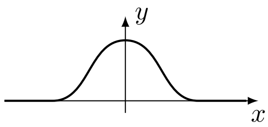
Problem 156. Suppose $f(x,y)$ is continuous on $[0,1]\times[0,1]$. Prove that $g(x) := \max_{y\in[0,1]} f(x,y)$ is also continuous.

Problem 157. Suppose f has continuous first and second order derivatives on $(-1,1)$ and $f''(x) \neq 0$ for all $x \in (-1,1)$.

(a) Show that, for all nonzero $x \in (-1,1)$, there exists a unique $\theta = \theta(x) \in (0,1)$ such that
$$f(x) = f(0) + xf'\big(x\theta(x)\big).$$

(b) Show that $\lim_{x\to 0} \theta(x) = \dfrac{1}{2}$.

Definition. Define $C_c(\mathbb{R}) = \{f \in C(\mathbb{R}) : \text{exists a compact set } K \text{ in } \mathbb{R},\ f \equiv 0 \text{ on } \mathbb{R}\setminus K\}$. For example,



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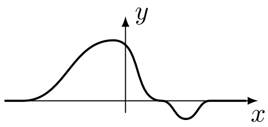

are functions in $C_c(\mathbb{R})$ (need not to share the same compact set). Such function is said to have compact support. We then define $C_c^+(\mathbb{R}) = \{f \in C_c(\mathbb{R}) : f \geq 0\}$.

Problem 158. Given $f, g \in C_c^+(\mathbb{R})$, $g \not\equiv 0$, show that there are $a_i > 0$ and $s_j \in \mathbb{R}$ such that
$$f(x) \leq \sum_{j=1}^{n} a_j g(x - s_j), \quad \forall x \in \mathbb{R}.$$

Problem 159. Let $f(x)$ be a function on the interval (a,b) that is strictly increasing and concave with continuous derivative. Suppose that for all $x \in (a,b)$ we have $a < f(x) < x$ and $\lim_{x\to a^+} f'(x) = 1$. Define $f_1(x) = f(x)$ and $f_n(x) = f_{n-1}(f(x))$ for $n \geq 2$, so $f_n(x)$ is the function f applied n-times on x.

Prove that for every $x \in (a,b)$
$$\lim_{n\to\infty} \frac{f_{n+2}(x) - f_{n+1}(x)}{f_{n+1}(x) - f_n(x)} = 1.$$

Problem 160. Let $f(x)$ and $g(x)$ be two differentiable functions such that
$$\frac{d}{dx} f(x) = -g(x) \quad \text{and} \quad \frac{d}{dx}\big(xg(x)\big) = xf(x).$$

(a) Show that between two consecutive roots of $f(x)$, $g(x)$ has a root.

(b) Show that between two consecutive roots of $g(x)$, $f(x)$ has a root.

Problem 161. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathbb{R}$. If $f(0) = f(1) = 0$ and $\max\{f(x) : x \in [0,1]\} = 2$, then prove that there exists $\theta \in (0,1)$ such that $f''(\theta) \leq -16$.

Problem 162. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and decreasing. Prove that there exists a unique element $(a,b,c) \in \mathbb{R}^3 = \mathbb{R}\times\mathbb{R}\times\mathbb{R}$ such that
$$a = f(b), \quad b = f(c) \quad \text{and} \quad c = f(a).$$

Problem 163. Show that every bijection $f : \mathbb{R} \to [0,+\infty)$ has infinitely many points of discontinuity.

Problem 164. Suppose that $f : \mathbb{R} - \{0,1\} \to \mathbb{R}$ satisfies the equation $f(x) + f\left(\dfrac{x-1}{x}\right) = 1 + x$, find $f(x)$.

Problem 165. Let $X \subset \mathbb{R}^m$ be compact (i.e. closed and bounded in Euclidean space). Let f be continuous and injective on X. Prove that $f^{-1} : f(X) \subset \mathbb{R}^n \to X$ is also continuous.

Remark. *There is a result in point-set topology which asserts that:*

> *If X is compact and Y is Hausdorff, then any bijective continuous map $f : X \to Y$ is a homeomorphism.*

Problem 166. Let $f : (0, \infty) \to \mathbb{R}$ be a uniformly continuous surjective map. Then prove that

(a) for any $a \in \mathbb{R}$, there are infinitely many b 's $\in (0, \infty)$ s.t. $f(b) = a$;

(b) such an f actually exists.

Problem 167. Let $f(x) = \left|1 - \dfrac{1}{x}\right|, x > 0$, if $0 < a < b$ and $f(a) = f(b)$, then prove that we must have $ab > 1$.

Problem 168. Let f be differentiable and $f'(a) < f'(b)$, $\forall y_0 \in (f'(a), f'(b))$, prove that $\exists c \in (a, b)$ such that $f'(c) = y_0$.
(**Note**: Differentiability of $f(x)$ cannot imply the continuity of $f'(x)$, Intermediate Value Theorem fails to work.)

Problem 169. Consider a continuous function f on $[a, b]$, suppose for any $a \leq x \leq b$, there is $a \leq y \leq b$, such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Prove that there is a $c \in [a, b]$ such that $f(c) = 0$.

Problem 170. Let $f, g : (a, b) \to \mathbb{R}$ be differentiable such that $f(x)g'(x) - f'(x)g(x) \neq 0$ for all $x \in (a, b)$. If there exist x_0, x_1 such that $a < x_0 < x_1 < b$ and $f(x_0) = f(x_1) = 0$, then prove that there exists $c \in (x_0, x_1)$ such that $g(c) = 0$.

Problem 171. For $f \in C^2(\mathbb{R})$ (i.e. f' and f'' exist and are continuous on $\mathbb{R}$), if f is bounded, then prove that there exists x_0 such that $f''(x_0) = 0$.

Problem 172. Prove that there does not exist a differentiable function f on $\mathbb{R}$ such that $f \circ f(x) = -x^3 + x^2 + 1$.

Problem 173. Suppose that $f : [0, 1] \to \mathbb{R}$ has continuous derivative and that $\displaystyle\int_0^1 f(x)\, dx = 0$. Prove that for every $\alpha \in (0, 1)$, $\left|\displaystyle\int_0^\alpha f(x)\, dx\right| \leq \dfrac{1}{8} \max_{0 \leq x \leq 1} |f'(x)|$.

Problem 174. Let $f : \mathbb{R} \to \mathbb{R}$, be a three times differentiable function. If $f(x)$ and $f'''(x)$ are bounded functions on $\mathbb{R}$, show that f' and f'' are also bounded functions on $\mathbb{R}$.

Problem 175. Let f be p times differentiable on $\mathbb{R}$ and let $M_k = \sup\{|f^{(k)}(x)| : x \in \mathbb{R}\} < \infty$, $k = 0, 1, 2, \ldots, p$ and $p \geq 2$. Prove that

(a) $M_1 \leq \sqrt{2M_0M_2}$

(b) $M_k \leq 2^{\frac{k(p-k)}{2}} M_0^{1-\frac{k}{p}} M_p^{\frac{k}{p}}$, for $k = 1, 2, \ldots, p - 1$.

Problem 176. Let f be continuous at $x = 0$, if $\displaystyle\lim_{x \to 0} \frac{f(2x) - f(x)}{x} = m$, prove that $f'(0) = m$.

Problem 177. Let $f : [0, \infty) \to \mathbb{R}$ with $f(0) = -1$ be a differentiable function so that $|f(x) - f'(x)| < 1$, $\forall x \geq 0$.

(a) Prove that f does have a limit that is infinite.

(b) Give an example of such a function.

Problem 178. Let $f : [0,1] \to [0,1]$, $g : [0,1] \to [0,1]$ be continuous and satisfy $f \circ g(x) = g \circ f(x)$. Prove that there is a $w \in [0,1]$ such that $f(w) = g(w)$.

Problem 179. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable and for all $x \in [0,1]$, $|f''(x)| \le 2010$. If there exists $c \in (0,1)$ such that $f(c) > f(0)$ and $f(c) > f(1)$, then prove that

$$|f'(0)| + |f'(1)| \le 2010.$$

Problem 180. $P(x)$ is a polynomial of degree n such that for all $w \in \{1, 2, 2^2, \ldots, 2^n\}$, we have $p(w) = 1/w$. Determine $P(0)$ with proof.

Problem 181. Let c be a real constant such that $\lim\limits_{x\to 0} \dfrac{f(x)}{x} = c$. Find $f(x)$ which satisfies $f(x+y) \le f(x) + f(y)$, for all real numbers x, y.

Problem 182. Let $\mathbb{Z}$ denote the set of all integers. Determine (with proof) all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all x, y in $\mathbb{Z}$, we have $f(x + f(y)) = f(x) - y$.

Problem 183. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a function in a neighborhood J of 0 s.t. $g(0) = 0$ and $g'(0) > 0$. Prove that if g has bounded second derivative in J, then the function $f(x) = \displaystyle\int_0^x g(x^2 t)\, dt$ has a local minimum at 0.

Problem 184. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous function such that

$$f\left(r + \frac{1}{n}\right) = f(r), \forall r \in \mathbb{Q}, \forall n \in \mathbb{N}.$$

Prove that f is constant function.

Problem 185. Let $f : [0,1] \to \mathbb{R}$ be continuous and $|f(x)| \le \displaystyle\int_0^x f(t)\, dt$, for all $x \in [0,1]$. Show that f is constantly zero on $[0,1]$.

Problem 186. Let P be a nonconstant polynomial with real coefficients and only real roots. Prove that for each $r \in \mathbb{R}$, the polynomial $Q_r(x) \triangleq P(x) - rP'(x)$ has only real roots.

Problem 187. Let $f : I \to \mathbb{R}$ be differentiable on the interval I. For a given $a \in I$, suppose for every sequences $\{x_n\}$, $\{y_n\}$ satisfying $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = a$ with $x_n \ne y_n$, one has $\lim\limits_{n\to\infty} \dfrac{f(y_n) - f(x_n)}{y_n - x_n} = f'(a)$. Prove that f' is continuous at a.

Problem 188. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. A point x is called a shadow point if there exists a point $y \in \mathbb{R}$ with $y > x$ such that $f(y) > f(x)$. Let $a < b$ be real numbers and suppose that

- All the points of the open interval $I = (a,b)$ are shadow points;
- a and b are not shadow points.

(a) Show that $f(x) \le f(b)$ for all $x \in (a,b)$.

(b) Show that $f(a) = f(b)$.

Problem 189. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function so that $\left|f(x) - \sin\left(x^2\right)\right| \le \dfrac{1}{4}$ for any $x \in \mathbb{R}$. Prove that there exists a sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ for which $\lim\limits_{n\to\infty} f'(x_n) = +\infty$.

1.7.2 Complex-valued Function

Problem 190. Show that if f is holomorphic on $\{z : |z| \leq 1\}$, then there must be some positive integer k such that

$$f\left(\frac{1}{k}\right) \neq \frac{1}{k+1}.$$

Problem 191. Suppose f is holomorphic on the annulus $\{z : 1 \leq |z| \leq 2\}$, $|f(z)| \leq 1$ for $|z| = 1$ and $|f(z)| \leq 4$ for $|z| = 2$. Prove that $|f(z)| \leq |z|^2$ throughout the annulus.

Problem 192. Let G be a open connected domain and $f : G \to \mathbb{C}$ be continuous. If f^2 is holomorphic on G, show that f is holomorphic on G. [Hint: First show f is holomorphic at z such that $f(z) \neq 0$, then consider the singularity type of roots of f.]

Problem 193. Let f be an entire function which is real on the real axis and imaginary on the imaginary axis, show that f is an odd function, i.e. $f(z) = -f(-z)$.

Problem 194. Suppose f is a nonconstant holomorphic function on the closed annulus $A = \{z : 1 \leq |z| \leq 2\}$. If f sends the boundary circles of A into the unit circle, show that f must have a root in A.

Problem 195. Suppose f and g are holomorphic on the closed unit disk. Show that $|f(z)| + |g(z)|$ takes its maximum on the boundary. [Hint: Consider $f(z)e^{i\alpha} + g(z)e^{i\beta}$ for appropriate α and β.]

Problem 196. Let $H = \{z : \operatorname{Re} z > 0\}$. Suppose $f : H \to H$ is holomorphic and $f(1) = 1$. Show that

$$|1 - f(2)| \leq \frac{1}{3}|1 + f(2)|.$$

Problem 197. Let f and g be holomorphic on a domain U. If $f\overline{g}$ is holomorphic on U, show that either $f \equiv 0$ or g is a constant function.

Problem 198. Find the maximum of $|f(\frac{1}{2})|$, where f is holomorphic on $D = \{z : |z| < 2\}$, $f(1) = 0$ and $|f(z)| \leq 10$, for $z \in D$.

Problem 199. Let D be the open unit disk. If $f : D \to D$ is holomorphic with at least two fixed points (i.e. points w such that $f(w) = w$), show that $f(z) \equiv z$. [Hint: By composing with a suitable Möbius mapping, one of the fixed points may be moved to the origin.]

Problem 200. Find all holomorphic function(s) f defined on the open unit disk $B(0,1)$ satisfying $f(\frac{1}{2}) = \frac{2}{3}$ and $f(z) = \big(2 - f(z)\big)f(2z)$, for all $z \in B(0,1)$.

Problem 201. Let w and z be in the open unit disk $B(0,1)$. If $f : B(0,1) \to B(0,1)$ is holomorphic and $f(w) = z$, prove that $|f'(w)| \leq \dfrac{1 - |z|^2}{1 - |w|^2}$.

Problem 202. If f is an entire function mapping the unit circle into the unit circle (i.e. $|f(z)| = 1$ for $|z| = 1$), show that $f(z) = e^{i\theta}z^n$ for some $\theta \in \mathbb{R}$ and some positive integer n. [Hint: consider roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ in the unit disk and recall that $\left|\dfrac{z - \alpha_j}{1 - \overline{\alpha_j}z}\right| = 1$ for $|z| = 1$. Show that $f(z) = e^{i\theta}\displaystyle\prod_{j=1}^{n}\frac{z - \alpha_j}{1 - \overline{\alpha_j}z}$ first.]

Problem 203. (Generalized Maximum Principle) Let $U \subset \mathbb{C}$ be a bounded domain and $f : U \to \mathbb{C}$ be holomorphic. Assume that for every sequence $z_n \in U$ which converges to the boundary of U, we have $\varlimsup_{n\to\infty} |f(z_n)| \leq M$. Prove that $|f(z)| \leq M$ for every $z \in U$.

Problem 204. (Generalized Schwarz Lemma) Let $f : B(0,1) \to B(0,1)$ be holomorphic and $f(z) = 0$ for $z = z_1, z_2, \ldots, z_n$. Show that for all $z \in B(0,1)$, $|f(z)| \leq \prod_{k=1}^{n} \left| \frac{z - z_k}{1 - \overline{z_k} z} \right|$.

Problem 205. Suppose f is an analytic function defined everywhere in $\mathbb{C}$ such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ is equal to 0. Prove that f is a polynomial.

Problem 206. Suppose D is simply connected and f is nonconstant holomorphic function on D. Show that there exists a holomorphic g on D such that $f = g^2$ if and only if every zero of f has even order.

Problem 207. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Suppose $\sum_{n=2}^{\infty} n|a_n| \leq 1$.

(a) Prove that f is holomorphic on the open unit disk D.

(b) Prove that f is injective on D. [Hint: use Rouches's theorem and consider the number of roots of $f(z) - f(z_0)$ for each $z_0 \in D$.]

Problem 208. Let n be a positive integer. For $i = 1, 2, \ldots, n$, let z_i and w_i be complex numbers such that for all 2^n choices of $\epsilon_1, \epsilon_2, \ldots, \epsilon_n \in \{1, -1\}$, we have

$$\left| \sum_{i=1}^{n} \epsilon_i z_i \right| \leq \left| \sum_{i=1}^{n} \epsilon_i w_i \right|.$$

Prove that

$$\sum_{i=1}^{n} |z_i|^2 \leq \sum_{i=1}^{n} |w_i|^2.$$

Problem 209. Let $\{f_n\}$ be a sequence of functions holomorphic on $B(0,1)$. Show that if $\{f_n\}$ converges uniformly on every compact subset of $B(0,1)$, so is $\{f_n'\}$.

1.8 Real Analysis

All functions here are extended real-valued. The set functions m and m^* denote Lebesgue measure and outer measure on $\mathbb{R}$ respectively. All integral "$\int_E$" on a measurable set E denotes Lebesgue integral over E. $\lambda(I)$ denotes the length of bounded interval I.

Problem 210. Show that if a set E has positive outer measure, then there is a bounded subset of E that also has positive outer measure.

Problem 211. Show that if E has finite measure and $\epsilon > 0$, then E is the disjoint union of a finite number of measurable sets, each of which has measure at most ϵ.

Problem 212. Show that a set is measurable if and only if for each $\epsilon > 0$, there is a closed set F and open set O for which $F \subseteq E \subseteq O$ and $m^*(O \setminus F) < \epsilon$.

Remark. *Recall the* ***Outer Measure Property****: E is measurable $\iff$ For each $\epsilon > 0$, there is an open set O containing E for which $m^*(O \setminus E) < \epsilon$ $\iff$ There is a G_δ set G containing E for which $m^*(G \setminus E) = 0$.*

And the ***Inner Measure Property****: E is measurable $\iff$ For each $\epsilon > 0$, there is a closed set F contained in E for which $m^*(E \setminus F) < \epsilon$ $\iff$ There is an F_σ set F contained in E for which $m^*(E \setminus F) = 0$.*

Problem 213. Suppose f and g are continuous functions on $[a, b]$. Show that if $f = g$ a.e. on $[a, b]$, then, in fact, $f = g$ on $[a, b]$. Is a similar assertion true if $[a, b]$ is replaced by a general measurable set E?

Problem 214. Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable subsets of $\mathbb{R}$ for which $\sum_{k=1}^{\infty} m(E_k) < \infty$.

(a) Show that

$$\{x \in \mathbb{R} : x \text{ lies in infinitely many of } A_k\text{'s}\} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \triangleq \overline{\lim_{n\to\infty}} E_n.$$

(b) Hence show that almost all $x \in \mathbb{R}$ belong to at most finitely many of the E_k's.

Problem 215 (Dini's theorem)**.** Let $\{f_n\}$ be an increasing sequence of continuous functions on $[a, b]$ which converges pointwise on $[a, b]$ to the continuous function f on $[a, b]$. Show that the convergence is uniform on $[a, b]$. [Hint: For $\epsilon > 0$ and for each natural number n, show that $\{E_n\}$ defined by $E_n = \{x \in [a, b] : f(x) - f_n(x) < \epsilon\}$ is an open cover of $[a, b]$.]

Problem 216. It is known that if f is measurable, then $f^{-1}(c)$ is measurable for any $c \in \mathbb{R}$ (if $c \notin$ range of f, then $f^{-1}(c) = \emptyset$). How about the converse? That is, suppose f is a function on $\mathbb{R}$ such that $f^{-1}(c)$ is measurable for each number $c \in \mathbb{R}$. Is f necessarily measurable?

Problem 217. Let I be a compact interval and E a measurable subset of I. Let $\epsilon > 0$, show that there is a step function h on I and a measurable subset F of I for which

$$h = \chi_E \text{ on } F \quad \text{and} \quad \mathrm{m}(I \setminus F) < \epsilon.$$

[Hint: Use the first principle.]

Problem 218. Let I be a compact interval and ψ a simple function defined on I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which

$$h = \psi \text{ on } F \quad \text{and} \quad m(I \setminus F) < \epsilon.$$

If $m \leq \psi \leq M$, then we can take h so that $m \leq h \leq M$. That is to say, each simple function on E is "nearly" a step function.

Problem 219. Let I be a compact interval and f a bounded measurable function defined on I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which

$$|f - h| < \epsilon \quad \text{and} \quad m(I \setminus F) < \epsilon.$$

[Recall that step function φ on $[a, b]$ has a canonical representation $\varphi = \sum_{i=1}^{n} a_i \chi_{I_i}$, where I_i are bounded interval.]

Problem 220. Let E have finite measure and f be a measurable function that is finite a.e.. Prove that given $\epsilon > 0$, there is a subset F of E such that

$$f \text{ is bounded on } F \quad \text{and} \quad m(E \setminus F) < \epsilon.$$

That is to say, each measurable function on a set of finite measure is "nearly" a bounded measurable function.

Definition. Let $\{f_n\}$ be a sequence of measurable functions on E and f a measurable function on E for which f and each f_n is finite a.e. on E. The sequence $\{f_n\}$ is said to **converge in measure on** E **to** f (denoted by $f_n \overset{m}{\to} f_n$) provided for each $\eta > 0$,

$$\lim_{n\to\infty} m\{x \in E : |f_n(x) - f(x)| > \eta\} = 0.$$

Problem 221. Let f_n $(n \geq 1)$, f and g be measurable functions on E that is finite a.e. on E. Assume $f_n \overset{m}{\to} f$, show that

$$f_n \overset{m}{\to} g \iff f = g \text{ a.e. on } E.$$

Problem 222. Let E have finite measure. Assume $f_n \overset{m}{\to} f$ and $g_n \overset{m}{\to} g$ on E, where f_n, g_n, f and g are measurable and finite a.e. on E. Prove that for every constants $\alpha, \beta \in \mathbb{R}$,

$$|f_n| \overset{m}{\to} |f|, \quad \alpha f_n + \beta g_n \overset{m}{\to} \alpha f + \beta g \quad \text{and} \quad f_n \cdot g_n \overset{m}{\to} f \cdot g.$$

Problem 223. Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E and f a measurable function on E for which f and each f_n is finite a.e. on E. Prove that

$$f_n \overset{m}{\to} f \iff \begin{array}{c}\text{Every subsequence of } \{f_n\} \text{ has in turn a further subsequence} \\ \text{that converges to } f \text{ pointwise a.e. on } E.\end{array}$$

Problem 224. Assume $m(E) < \infty$. For two measurable functions g and h on E, define $\rho(g, h) = \displaystyle\int_E \frac{|g-h|}{1+|g-h|}$. Show that

$$f_n \overset{m}{\to} f \iff \lim_{n\to\infty} \rho(f_n, f) = 0.$$

Problem 225. Let f be a bounded measurable function on $E \subseteq \mathbb{R}$. Assume that there are constants $C > 0$ and $0 < \alpha < 1$ such that

$$m\{x \in E : |f(x)| > \epsilon\} < \frac{C}{\epsilon^\alpha}$$

for every $\epsilon > 0$. Show that $\int_E |f| < \infty$.

Problem 226. Let E be a measurable subset of $\mathbb{R}$, $m(E) < \infty$ and $\{f_n\}$ be a sequence of measurable functions on E. Let $\{\alpha_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^\infty m\{x \in E : |f_n(x)| > \alpha_n\} < \infty$. Prove that

$$-1 \leq \varliminf_{n\to\infty} \frac{f_n(x)}{\alpha_n} \leq \varlimsup_{n\to\infty} \frac{f_n(x)}{\alpha_n} \leq 1$$

for almost all $x \in E$.

Problem 227. Let $f(x)$ be a positive integrable function on $[a, b]$, $\{E_n\}$ a collection of measurable subsets of $[a, b]$. Show that

$$\lim_{n\to\infty} \int_{E_n} f(x) = 0 \implies \lim_{n\to\infty} m(E_n) = 0.$$

Problem 228 (General Lebesgue Dominated Convergence Theorem)**.** Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f. Suppose there is a sequence $\{g_n\}$ of nonnegative measurable functions on E that converges pointwise a.e. on E to g and dominates $\{f_n\}$ on E in the sense that

$$|f_n| \leq g_n \text{ on } E \text{ for all } n.$$

Show that

$$\lim_{n\to\infty} \int_E g_n = \int_E g < \infty \implies \lim_{n\to\infty} \int_E f_n = \int_E f.$$

[hint: just imitate the proof of Lebesgue dominated convergence theorem, how can we apply Fatou's lemma?]

Problem 229. Let f be integrable (tacitly assumed measurable) over $\mathbb{R}$ and $\epsilon > 0$. Establish the following three approximation properties.

(a) There is a simple function η on $\mathbb{R}$ which has finite support and $\int_{\mathbb{R}} |f - \eta| < \epsilon$. (do it for non-negative function first)

(b) There is a step function s on $\mathbb{R}$ which vanishes outside a closed, bounded interval and $\int_{\mathbb{R}} |f - s| < \epsilon$.

(c) There is a continuous function g on $\mathbb{R}$ which vanishes outside a bounded set and $\int_{\mathbb{R}} |f - g| < \epsilon$.

Remark. *Now the result can be extended to integration over any measurable subset of* $\mathbb{R}$.

Problem 230. Let f be integrable over $(-\infty, \infty)$.

(a) Show that for each t,
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x+t)\,dx.$$
[hint: density of step functions]

(b) Let g be a bounded measurable function on $\mathbb{R}$. Show that
$$\lim_{t\to 0} \int_{-\infty}^{\infty} g(x) \cdot \big(f(x) - f(x+t)\big)\,dx = 0.$$
[hint: density of continuous functions]

Problem 231. Show that a set E of real numbers has measure zero if and only if there is a countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ for which each point in E belongs to infinitely many of the I_k's and $\sum_{k=1}^{\infty} \lambda(I_k) < \infty$.

Problem 232 (Riesz-Nagy). Let E be a set of measure zero contained in the open interval (a, b). According to the preceding problem, there is a countable collection of open tervals contained in (a, b), $\{(c_k, d_k)\}_{k=1}^{\infty}$, for which each point in E belongs to infinitely many intervals in the collection and $\sum_{k=1}^{\infty}(d_k - c_k) < \infty$. Define
$$f(x) = \sum_{k=1}^{\infty} \lambda\big((c_k, d_k) \cap (-\infty, x)\big)$$
for all $x \in (a, b)$. Show that f is increasing and fails to be differentiable at each point in E.

Problem 233. Let f be of bounded variation on $[a, b]$ and define $v(x) = TV(f_{[a,x]})$ for all $x \in [a, b]$.

(a) Show that $|f'| \le v'$ a.e. on $[a, b]$, and infer from this that
$$\int_a^b |f'| \le TV(f).$$

(b) Show that the above is an equality if and only if f is absolutely continuous on $[a, b]$.

Problem 234. Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz, that is, there is a constant L such that $|f(x) - f(y)| \le L|x - y|$ for any $x, y \in \mathbb{R}$, prove that for any $A \subseteq \mathbb{R}$, one has
$$m^*\big(f(A)\big) \le L m^*(A).$$
Then prove that a Lipschitz function takes bounded measurable subsets to bounded measurable subsets, does it take any measurable set to measurable set?

Problem 235. Let $f, g > 0$ be integrable extended real-valued functions on measurable E such that $fg \geq 1$ and $m(E) = 1$. Prove that

$$\int_E f \int_E g \geq 1.$$

Problem 236. Complete the proof of Lusin's theorem.

Problem 237. Suppose f is Lebesgue integrable function on $\mathbb{R}$, prove that

$$\lim_{n\to+\infty} \int_{\mathbb{R}} f(x) \cos nx \, dm(x) = \lim_{n\to\infty} \int_{\mathbb{R}} f(x) \sin nx \, dm(x) = 0.$$

And in general, if g is bounded integrable function with period $T > 0$, then

$$\lim_{t\to+\infty} \int_{\mathbb{R}} f(x) g(tx) \, dm(x) = \frac{1}{T} \int_{[0,T]} g(x) \, dm \int_{\mathbb{R}} f(x) \, dm.$$

Problem 238. Let $f : [a, b] \to \mathbb{R}$ be bounded and continuous m-a.e.[2] on $[a.b]$, here m denotes Lebesgue measure on $\mathbb{R}$.

(a) Let $\{P_n\}_{n\geq 1}$ be any sequence of partitions of $[a, b]$ such that each P_{n+1} refines P_n and $\|P_n\| \to 0$. Let φ_n and ψ_n ($\varphi_n \leq f \leq \psi_n$) be defined as in theorem **??**. Let $x \in (a, b)$ be a point of continuity of f, show that

$$\lim_{n\to\infty} \varphi_n(x) = f(x) = \lim_{n\to\infty} \psi_n(x).$$

(b) Using (a) and the dominated convergence theorem, deduce that

$$\int_{[a,b]} f \, dm = \lim_{n\to\infty} \int_{[a,b]} \varphi_n \, dm = \lim_{n\to\infty} \int_{[a,b]} \psi_n \, dm.$$

(c) Show that f is Riemann integrable on $[a, b]$ and

$$\int_{[a,b]} f \, dm = \int_a^b f(x) \, dx.$$

1.9 Fourier Analysis

- Let both $\int_Q \bullet$, $\int_Q \bullet(x) \, dx$ denote the Lebesgue integral over Q.
- For any two functions $f, g \in L^2(Q)$, we denote the inner product of f and g by $(f, g) = \int_Q f\overline{g}$.
- For $n \in \mathbb{N}$ and an orthonormal collection $\{e_n(x)\}_{n=1}^{\infty}$ on Q, $\hat{f}(n) := (f, e_n) = \int_Q f\overline{e_n}$. We say $\{e_n\}$ is complete or a basis of $L^2(Q)$ if for any $f \in L^2(Q)$, $f = \sum_{n=1}^{\infty} \hat{f}(n) e_n$ in the sense of L^2 distance.
- Any function in this section is complex-valued function on $Q \subseteq \mathbb{R}$ whose real and imaginary part are both measurable functions.
- A collection $\{e_n\}$ is indexed by $n \in \mathbb{N}$ or $n \in \mathbb{Z}$ when one of them is convenient.
- For two vectors (or sequences) $a = (a_1, a_2, a_3, \dots), b = (b_1, b_2, b_3, \dots) \in \ell^2$, we define $(a, b) = \sum_{n\geq 1} a_n \overline{b_n}$.

[2]Some property P holds μ-a.e. means P holds except a set of μ-measure zero.

- Define the "circle" to be $S^1 = \mathbb{R}/\mathbb{Z}$, i.e. $[0,1)$ with 0 and 1 identified. Henceforth functions defined on S^1 are 1-periodic functions (another useful convention is $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$).

Problem 239. Check that for any orthonormal collection and any $f \in L^2(Q)$,

$$\sum_{k=1}^{\infty} |\hat{f}(n)|^2 \leq \|f\|^2$$

and conclude that $\{e_n\}_{n=1}^{\infty}$ is a basis if and only if the Plancherel identity holds:

$$\sum_{k=1}^{\infty} |\hat{f}(n)|^2 = \|f\|^2$$

for every $f \in L^2(Q)$.

Problem 240. Let $e_n : n \geq 1$ be any orthonormal collection in $L^2(Q)$. Then for any $f \in L^2(Q)$, any $n \geq 1$, and any complex numbers $c_1, \ldots, c_n$,

$$\left\| f - \sum_{k=1}^{n} \hat{f}(k) e_k \right\| \leq \left\| f - \sum_{k=1}^{n} c_k e_k \right\|.$$

The lower bound on the LHS is attained if and only if $c_k = \hat{f}(k), \forall k \leq n$.

Problem 241. Show that $L^2(Q)$ is infinite dimensional.

Problem 242. Check that the map $f \mapsto \hat{f} = (\hat{f}(1), \hat{f}(2), \hat{f}(3), \ldots)$ preserves inner products:

$$(f_1, f_2) = \int_Q f_1 \overline{f_2} = (\hat{f}_1, \hat{f}_2) = \sum_{n=1}^{\infty} \hat{f}_1(n) \overline{\hat{f}_2(n)}.$$

Definition (for problem 243 to 248)**.** Let $A \subseteq L^2(Q)$ be a closed subspace, by closed we mean limit points of A are still in A. Define the **annihilator** of A, $A^\perp$, to be the class of functions from $L^2(Q)$ that are "perpendicular" to every function from A.

Problem 243. Check that $A^\perp$ is a closed subspace, what if A is not closed?

Problem 244. For any $f \in L^2(Q)$, there is a point Pf in A which is closest to f, i.e.

$$\|f - Pf\| \leq \|f - g\|, \quad \forall g \in A.$$

[Hint: Pick $g_n \in A$ so as to make $\lim_{n\to\infty} \|f - g_n\| = \inf_{g \in A} \|f - g\|$. Then apply the Gram-Schmidt recipe to convert $g_n : n \geq 1$ into an orthonormal sequence $e_n : n \geq 1$ and put $Pf = \sum (f, e_n) e_n$.]

Problem 245. Prove that $f - Pf \in A^\perp$ and that $f = Pf + (f - Pf)$ is the only way of splitting f into a piece from A and a piece from B. That is,

$$L^2(Q) = A \oplus A^\perp.$$

[Hint: Pick $g \in A$, then $k(\epsilon) := \|f - Pf + \epsilon g\|^2$ is a polynomial of degree 2, consider its least value.]

Problem 246. The so-called projection $f \mapsto Pf$ in problem 244 is a linear map of $L^2(Q)$ into itself, check it. Besides, verify that:

(i) $P^2 = P$;

(ii) $(Pf_1, f_2) = (f_1, Pf_2)$;

(iii) $\|Pf\| \le \|f\|$;

(vi) $P = \text{id}$ on A and 0 on $A^\perp$.

Problem 247. Show that the family $f_n : n \ge 1$ spans $L^2(Q)$ if and only if $(f, f_n) = 0$ for all $n \ge 1$ implies $f \equiv 0$.

Problem 248. Show that any linear map T of $L^2(Q)$ into the complex numbers which is bounded in the sense that

$$T(f) \le \text{constant} \times \|f\|,$$

where the constant is independent of f, can be expressed as an inner product:

$$T(f) = (f, g)$$

for some $g \in L^2(Q)$. This is the so-called Riesz representation theorem.
[Hint: Suppose $T \not\equiv 0$ and let $A = \ker T$. Check that $A^\perp$ is of dimension 1 and find a function $g \in L^2(Q)$ so that $T(f) = 0 \iff (f, g) = 0$. Then

$$T(f) = \|g\|^{-2} T(g)(f, g) = \text{constant} \times (f, g).$$

]

Problem 249. The orthonormal collection

$$e_n(x) = e^{2\pi i n x}, \quad n \in \mathbb{Z}$$

is a basis for $L^2(S^1)$, that is, any function $f \in L^2(S^1)$ can be expanded into a fourier series

$$f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n$$

while the limit is taken in the sense of L^2 distance, with coefficients

$$\hat{f}(n) = (f, e_n) = \int_0^1 f \overline{e_n} = \int_0^1 f(x) e^{-2\pi i n x}\, dx.$$

[Hint: The space $C^1(S^1)$ is dense in $L^2(S^1)$.]

Remark. *Hence for any function $f \in L^2(Q)$, completeness of $\{e^{2\pi i n x}\}_{n \in \mathbb{Z}}$ implies we always have the Plancherel identity:*

$$\|f\|^2 = \int_0^1 |f|^2 = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 = \|\hat{f}\|.$$

Definition (for all of later problems)**.** Until further notice, the orthonormal collection e_n will always mean $e^{2\pi i n x}$ and "Fourier series" will refer to this particular collection.

Problem 250. Check that $f \in C^\infty(S^1)$ if and only if $\hat{f}$ is rapidly decreasing in the sense that $n^p \hat{f}(n)$ approaches 0 as $|n| \to +\infty$, for every $p < \infty$, separately.
[Hint: For rapidly decreasing $\hat{f}$, $\sum \hat{f}(n) e_n$ converges uniformly to a periodic function f_1, and

$$\int_0^x f_1 = \sum \hat{f}(n) \int_0^x e_n' = \sum \hat{f}(n)[e_n(x) - e_n(0)] = f(x) - f(0).$$

]

Problem 251. Show that any linear map $T : L^1(Q) \to \mathbb{C}$ subject to $|T(f)| \leq \text{constant} \times \|f\|$, with a constant independent of f, can be expressed as $T(f) = \int_Q f\overline{g}$ for some bounded measurable function g.
[Hint: $L^2(Q) \subseteq L^1(Q)$ if Q is bounded. Now apply problem 248 to find such a function $g \in L^2(Q)$ and check that

$$\int_a^b |g| \leq \text{constant} \times (b-a),$$

for any interval $a \leq x \leq b$.]

Definition. For $f, g \in L^1(S^1)$, we define **convolution** of f and g, denoted by $f * g$, to be the "product"

$$f * g = \int_0^1 f(x-y)g(y)\,dy.$$

Problem 252. Check that the convolution defined above makes sense, that is, check that $f * g \in L^1(S^1)$.

Problem 253. Check that $L^2(S^1)$ is an ideal in $L^1(S^1)$. This means that $f * g \in L^2(S^1)$ as soon as one of the factors does.

Problem 254. Check that $L^1(S^1)$ does not have a multiplicative identity.
[Hint: A multiplicative identity e would satisfy $e * f = f$. Now look at $\hat{e}(n)$ keeping the Riemann-Lebesgue lemma in mind.]

1.10 Number Theory

Let $\phi(k)$ be the number of positive integers less than or equal to k that are relatively prime to k, i.e. the Euler-ϕ function.

Problem 255. Let n be an odd number greater than 1, let $a_1, a_2, \ldots, a_{\phi(n)}$ be a reduced residue system modulo n (all a_i's are relatively prime to n), prove that $\left|\prod_{k=1}^{\phi(n)} \cos\frac{a_k\pi}{n}\right| = \frac{1}{2^{\phi(n)}}$.

Problem 256. Let $a, b \in \mathbb{N}$, show that if $4ab-1|(4a^2-1)^2$, then $a=b$.

Problem 257. Let $x, y \in \mathbb{N}$, find all the integral solution of $y^2 = x^3 + 7$.

Problem 258. For every positive integer n, let $a_n = 2^n + 3^n + 6^n - 1$. Prove that for every prime number $p \geq 5$, there exists a positive integer n such that p divides a_n (this can be done by Fermat's little theorem).

Problem 259. Let m and n be positive integers such that $m\phi(m) = n\phi(n)$, then prove that $m = n$. (**Note**: $\phi(n)$ itself is oscillating)

Problem 260. Prove that there is a bijection $f : \mathbb{N} \to \mathbb{N}$ such that for every positive integer k, $f(1) + f(2) + \cdots + f(k)$ is divisible by k (this can be done by the Chinese remainder theorem).

Problem 261. Determine all positive integers x and y such that $2x^4 + 1 = y^2$.

Problem 262. Determine all positive integers n such that $n^2 - 1$ divides $2^{n!} - 1$.

Problem 263. Let $x, y \in \mathbb{N}$, find all the ordered pair(s) (x, y) satisfying

$$y^2 - (x+1)2^x = 1$$

with proof. (hint given: There is only one solution)

Problem 264. Find all values of n such that $\phi(n) = \dfrac{n}{3}$.

Definition. Let $(a, m) = 1$, denote $\text{ord}_m(a)$ the **smallest integer** k such that $a^k \equiv 1 \pmod m$. Such k must exist as at least $a^{\phi(m)} \equiv 1 \pmod m$. If further $\text{ord}_m(a) = \phi(m)$ (i.e. $a^{\phi(m)} \equiv 1 \pmod m$, $\phi(m)$ is least possible), we call a **the primitive root of** m.

Some basic fact we have already known:

- $\text{ord}_m(a) = p \implies \text{ord}_m(a^q) = p/(p, q)$.
- If $a^h \equiv 1 \pmod m$, then $\text{ord}_m(a) \,|\, h$.
- If g is a primitive root, then $\{[g], [g^2], \ldots, [g^{\phi(m)}]\} = (\mathbb{Z}/m\mathbb{Z})^\times$ (the complete reduced residue class), here $[k] = k * (m\mathbb{Z}) = k + m\mathbb{Z}$, the coset of $m\mathbb{Z}$.

Problem 265. Let p be an odd prime. Prove that $\text{ord}_p(a) = 2$ if and only if $a \equiv -1 \pmod p$.

Problem 266. (a) If $\text{ord}_m(a) = h$, prove that no two of $a, a^2, \ldots, a^h$ are congruent modulo m.

(b) Let g be a primitive root modulo p, p is an odd prime. Show that (by using the known fact)

$$(p-1)! \equiv g \cdot g^2 \cdots g^{p-1} \equiv g^{\frac{p(p-1)}{2}} \pmod p.$$

Use this to give a proof of Wilson's congruence that

$$(p-1)! \equiv -1 \pmod p.$$

(c) Show that if g and g' are primitive roots modulo an odd prime p, then gg' is not a primitive root of p (you can use other fact that is not given in this question).

Problem 267. If $\gcd\big(\text{ord}_m(a), \text{ord}_m(b)\big) = 1$, show that

$$\text{ord}_m(ab) = \text{ord}_m(a)\,\text{ord}_m(b)\,.$$

Remark. *If "co-primeness" is dropped, one can argue that*

$$\text{ord}_m(ab) \Big| [h, k] = \frac{hk}{(h, k)}\,.$$

Inspired from the fact that $\text{ord}_m(a) = p \implies \text{ord}_m(a^q) = p/(p, q)$ *(here* $pq/(p, q)$ *is LCM!), one may conjecture "|" above can be replaced by "=". Unfortunately equality cannot hold in general.*

Problem 268. Give an example that $\text{ord}_m(ab) \neq [\text{ord}_m(a), \text{ord}_m(b)]$.

Problem 269. Let D not be a perfect square. Assume that $x^2 - Dy^2 = -1$ has integer solution, and let x_1, y_1 be its smallest positive solution. Prove that

(a) x_2, y_2 defined by

$$x_2 + y_2\sqrt{D} = (x_1 + y_1\sqrt{D})^2$$

is the smallest positive integer solution of $x^2 - Dy^2 = 1$.

(b) All solutions of $x^2 - Dy^2 = -1$ are given by (x_n, y_n), where

$$x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n, \quad n = 1, 3, 5, \ldots,$$

and that all solutions of $x^2 - Dy^2 = 1$ are given by (x_n, y_n), with $n = 2, 4, 6, \ldots$.

Problem 270. $n^2 + (n+1)^2$ is a perfect square for infinitely many values of n.

Problem 271. Let n be a natural number such that the equation $a^n + b^n = c^2$, where a, b and c are prime numbers, has at least one solution. Find the maximal possible value of n.

1.11 Metric Spaces

Problem 272. Show that a sequence is totally bounded in (S,d) if and only if every sequence in S has a Cauchy subsequence.

Problem 273. Let $f:\mathbb{R}\to\mathbb{R}$ be infinitely differentiable. If for every $w\in\mathbb{R}$, there exists a positive integer k such that the k-th derivative $f^{(k)}(w)=0$, then prove that on some non-empty open interval (a,b), f is a polynomial.

Definition. A F_σ set in M is a union of a countable number of closed sets in M. A G_δ set in M is an intersection of a coutable number of open set in M.

Problem 274. For $\emptyset\subset A\subseteq M$, define $d(x,A)=\inf\{d(x,y):y\in A\}$. Prove that $d(x,A)=0$ if and only if $x\in\overline{A}$. Prove that every closed set in M is a G_δ set in M. (Then by de Morgan's law, every open set in M is a F_σ set in M.)

Problem 275. (a) Let $C[0,1]$ denote the set of all continuous real-valued function on $[0,1]$. For $f,g\in C[0,1]$, define $d_\infty(f,g)=\sup\{|f(x)-g(x)|:x\in[0,1]\}$. Prove that d_∞ is a metric for $C[0,1]$.

(b) Prove that $C[0,1]$ is complete with sup-norm $\|f\|_\infty=\sup\{|f(x)|:x\in[0,1]\}$ by checking the completeness criterion.

(c) Prove that there is a unique continuous real-valued function $f(x)$ on $[0,1]$ such that

$$f(x)=\sin x+\int_0^1\frac{f(y)}{e^{x+y+1}}\,dy.$$

Problem 276. Prove the *Lebesgue Covering Theorem*: Let M be a compact metric space and $\mathcal{U}$ be an open covering of M. Prove that there exists $r>0$ such that for every $x\in M$, there is at least one $U\in\mathcal{U}$ satisfying $B(x,r)\subseteq U$. The constant r is called a *Lebesgue number* for the covering $\mathcal{U}$.

Problem 277. Let X be a compact metric space with d as the metric. If $f:X\to X$ satisfies $d\big(f(x),f(y)\big)<d(x,y)$ for all distinct $x,y\in X$, then prove that f has a fixed point.

Problem 278. Let $W_1,W_2,W_3,\dots$ be closed sets in $\mathbb{R}$ and $W_1^\circ,W_2^\circ,W_3^\circ,\dots$ be their interiors in $\mathbb{R}$ respectively. If $\mathbb{R}=W_1\cup W_2\cup W_3\cup\dots$, then prove that $S=W_1^\circ\cup W_2^\circ\cup W_3^\circ\cup\cdots$ is dense in $\mathbb{R}$.

Definition. Let f be a real (or extended-real) valued function on a metric space X. If

$$\{x\in X:f(x)>\alpha\}$$

is open for every real α, f is said to be **lower semicontinuous**.

Remark. *When X is any topological space, the notion of lower semicontinuity is defined in the same way. The simple example for such a function is the characteristic function of a open set in X.*

Problem 279. Suppose that X is a metric space, with metric d, and that $f:X\to[0,\infty]$ is lower semicontinuous, $f(p)<\infty$ for at least one $p\in X$. For $n=1,2,3,\dots$ and $x\in X$, define

$$g_n(x)=\inf\{f(p)+nd(x,p):p\in X\}.$$

Prove that:

(i) $|g_n(x)-g_n(y)|\le nd(x,y)$;

(ii) $0\le g_1\le g_2\le\cdots\le f$ and

(iii) $\lim_{n\to\infty} g_n(x) = f(x)$, for all $x \in X$.

Definition. Let X be a metric space and Λ a set of real numbers. A collection of open subsets of X $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ is said to be **normally ascending** provided for any $\lambda_1, \lambda_2 \in \Lambda$,

$$\overline{\mathcal{O}_{\lambda_1}} \subseteq \mathcal{O}_{\lambda_2} \text{ when } \lambda_1 < \lambda_2.$$

Problem 280. Let Λ be a dense subset of (a, b), where $a, b \in \mathbb{R}$, and $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ a normally ascending collection of open subsets of a metric space X. Define the function $f : X \to \mathbb{R}$ by setting $f = b$ on $X \setminus \cup_{\lambda\in\Lambda}\mathcal{O}_\lambda$ and otherwise setting

$$f(x) = \inf\{\lambda \in \Lambda : x \in \mathcal{O}_\lambda\}.$$

Show that $f : X \to [a, b]$ is continuous.

1.12 Linear Algebra

Denote $\mathcal{L}(U, V)$ a collection of linear maps from U to V. Define $\mathcal{L}(V) = \mathcal{L}(V, V)$. Here V is a **finite dimensional** vector space (unless otherwise specified) over $\mathbb{F}$ ($\mathbb{F}$ is $\mathbb{C}$ or $\mathbb{R}$).

Problem 281. Suppose $T \in \mathcal{L}(V)$ and $\dim \operatorname{range} T = k$. Prove that T has at most $k + 1$ distinct eigenvalues.

Problem 282. Suppose $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

Problem 283. Suppose $T \in \mathcal{L}(V)$ is such that every vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

Problem 284. Let $S, T \in \mathcal{L}(V)$, prove that $ST = I$ if and only if $TS = I$.

Problem 285. Let $S, T \in \mathcal{L}(V)$, prove that T is a scalar multiple of the identity if and only if $ST = TS$, for every $S \in \mathcal{L}(V)$.

Problem 286. Suppose $u, v \in V$. Prove that $\langle u, v\rangle = 0$ if and only if

$$\|u\| \le \|u + av\|, \text{ for all } a \in \mathbb{F}.$$

Definition. Let $T \in \mathcal{L}(V)$. Let U be a subspace of V. We say that U is invariant under T or U is an invariant subspace of T if and only if for every $u \in U$, $Tu \in U$. In short, $T|_U \in \mathcal{L}(U)$.

Problem 287. Suppose $T \in \mathcal{L}(V)$ is such that every subspace of V with dimension $\dim V - 1$ is invariant under T. Prove that T is a scalar multiple of the identity operator.

Problem 288. Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$.

Problem 289. Let U and V be finite dimensional vector spaces, $S \in \mathcal{L}(V, W)$, $T \in \mathcal{L}(U, V)$, prove that

$$\dim \operatorname{null} ST \le \dim \operatorname{null} S + \dim \operatorname{null} T.$$

Problem 290. Prove that if $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and

$$\|Pv\| \le \|v\|$$

for every $v \in V$, then P is an orthogonal projection.

Problem 291. Suppose V is a real inner-product space and $(v_1, ..., v_m)$ is a linearly independent list of vectors in V. Prove that there exist exactly 2^m orthonormal lists $(e_1, ..., e_m)$ of vectors in V such that

$$\operatorname{span}(v_1, \ldots, v_j) = \operatorname{span}(e_1, \ldots, e_j)$$

for all $j \in \{1, ..., m\}$.

Problem 292. Find a polynomial $q \in P_2(\mathbb{R})$ (a collection of polynomial of degree two with real coefficients) such that

$$\int_0^1 p(x)\cos \pi x \, dx = \int_0^1 p(x)q(x)\, dx$$

for every $p \in P_2(\mathbb{R})$.

Definition. Let T^* denote the adjoint of T.

Problem 293. Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Prove that λ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^*.

Problem 294. (a) Prove that if $\dim \operatorname{range} T = \dim \operatorname{range} T^2$, then $\operatorname{range} T \cap \operatorname{null} T = \{0\}$. Prove also that $V = \operatorname{null} T \oplus \operatorname{range} T$.

(b) Prove that for any $T \in \mathcal{L}(V)$, there is a positive integer k such that

$$V = \operatorname{range} T^k \oplus \operatorname{null} T^k.$$

Problem 295. Let A be an $m \times n$ matrix. Show that $\operatorname{rank} QAP = \operatorname{rank} A$ for any invertible $m \times m$ matrix Q and any invertible $n \times n$ matrix P.

Problem 296. Let A be an $m \times n$ matrix. Show that $\operatorname{rank} A^T A = \operatorname{rank} A$.

Problem 297. Let S be a skew-symmetric matrix (i.e. $S^T = -S$), prove that $I + S$ is invertible.

Definition. For an $n \times n$ matrix A (or an operator $T \in \mathcal{L}(V)$) we define the **spectrum of** A, as follows:

$$\sigma(A) := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}.$$

We also define $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ to be the **spectral radius of** A.

Problem 298. Let A be an $n \times n$ matrix, then

$$\lim_{n\to\infty} A^n = 0 \iff \rho(A) < 1.$$

Here $\lim_{n\to\infty} A^n = 0$ means that $\lim_{n\to\infty} A^n \mathbf{x} = 0$ for each $\mathbf{x} \in \mathbb{C}^n$.

Problem 299. Let A, B be Hermitian matrices that commute. Prove that there is a unitary matrix P such that P^*AP and P^*BP are both diagonal.

Problem 300. Define $A = (a_{ij})_{1\le i,j\le n}$ and $A_k = (a_{ij})_{1\le i,j\le k}$, $a_{ij} \in \mathbb{R}$ and $A^T = A$. Prove that A is positive definite if and only if $\det A_k > 0$ for $k = 1, 2, \ldots, n$.

Problem 301. Let $b \in \mathbb{C}^m$ and $A \in \mathbb{C}^{m\times n}$ be fixed. The following are equivalent:

(i) x_0 solves the least square problem (LSP) in the sense that

$$\|b - Ax_0\|_2 = \inf\{\|b - Ax\|_2 : x \in \mathbb{C}^n\}.$$

(ii) $b - Ax_0 \in (\operatorname{range} A)^\perp$.

(iii) $A^*Ax_0 = A^*b$.

Moreover, such x_0 is unique $\iff$ A has full rank.

Definition. Let X, Y be normed vector space, that is, X and Y are endowed with the norms $\|\cdot\|_X, \|\cdot\|_Y$ respectively. Let $L(X,Y)$ denote the collection of all continuous linear maps from X to Y. For each $T \in L(X,Y)$, we can define

$$\|T\| = \sup\{\|Tx\|_Y : x \in X, \|x\|_X = 1\},$$

$\|\cdot\|$ defined above turns out to be a norm on $L(X,Y)$, called **operator norm**. When X and Y are Euclidean spaces, $L(X,Y)$ is the collection of matrices, and the operator norm in this special case is called **induced matrix norm**.

Let $x \in \mathbb{F}^n$, write x as $(x^1, x^2, \dots, x^n)$, i.e., x^i denotes its ith component. This notation is not ambiguous as long as x is a vector. Recall that $\|x\|_p = (\sum_{i=1}^n |x^i|^p)^{1/p}$, $\|x\|_\infty = \max\{|x^i| : i = 1, 2, \dots, n\}$ and $x^* = \overline{(x^T)}$.

For each $p \geq 1$ and a matrix A over some scalar field, it is a convention to define

$$\|A\|_p := \sup\{\|Ax\|_p : x \text{ in domain}, \|x\|_p = 1\}.$$

Clearly it is a special case of operator norm, with the norms of domain and range being fixed to be p-norm. It is also a convention to denote $\mathbb{R}^{m\times n}$ and $\mathbb{C}^{m\times n}$ the collection of $m \times n$ real and complex matrices respectively.

Problem 302. Let a_i's be column vectors, show that:

(a) If $A = \left[\, a_1 \;\middle|\; \cdots \;\middle|\; a_n \,\right]$, $\|A\|_1 = \max_{1\leq j\leq n} \|a_j\|_1$;

(b) If $A = \begin{bmatrix} a_1^* \\ \hline \vdots \\ \hline a_n^* \end{bmatrix}$, then $\|A\|_\infty = \max_{1\leq i\leq m} \|a_i^*\|_1$.

In words, $\|A\|_1$ is the maximum (absolute) column sum, while $\|A\|_\infty$ is the maximum (absolute) row sum.

Problem 303. Let D be the diagonal matrix

$$D := \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix},$$

show that $\|D\|_2 = \max\{|d_i| : i = 1, 2, \dots, n\}$.

Problem 304. Vector and matrix p-norms are related by various inequalities, often involving dimensions m or n. For each of the following, verify the inequality and give an example of a nonzero vector of matrix (for general m, n) for which equality is achieved (so that the bound in optimal). In the problem x is an m-vector and A is an $m \times n$ matrix.

(a) $\|x\|_\infty \leq \|x\|_2$

(b) $\|x\|_2 \leq \sqrt{m}\|x\|_\infty$

(c) $\|A\|_\infty \leq \sqrt{n}\|A\|_2$

(d) $\|A\|_2 \leq \sqrt{m}\|A\|_\infty$

Problem 305. Let A be an $m \times n$ matrix and let B be a submatrix of A, that is, a $\mu \times \nu$ matrix ($\mu \leq m, \nu \leq n$) obtained by selecting certain rows and columns of A (not necessarily consecutive rows and columns!)

(a) Explain how B can be obtained by multiplying A by certain matrices.

(b) Using the result in (a), show that $\|B\|_p \leq \|A\|_p$ for any p with $1 \leq p \leq \infty$.

Problem 306. In this problem we are going to prove part (i) of the following theorem. Whereas part (ii) (which we don't go through in this problem!) requires the definition of singular vectors, which motivates part (i) of the theorem.

> **Theorem.**
>
> *(i) Every matrix $A \in \mathbb{C}^{m\times n}$ has a SVD:*
>
> $$A = U\Sigma V^* \qquad \begin{array}{|ll|} \hline U \in \mathbb{C}^{m\times m} & \textit{is unitary} \\ V \in \mathbb{C}^{n\times n} & \textit{is unitary} \\ \Sigma \in \mathbb{R}^{m\times n} & \textit{is "diagonal"} \\ \hline \end{array}$$
>
> *Furthermore, the singular values σ_j's, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min\{m,n\}} \geq 0$, are uniquely determined.*
>
> *(ii) If A is square and σ_j's are distinct, the left and right singular vectors $\{u_j\}$ and $\{v_j\}$ are unique up to a multiplicative constant with modulus 1.*

The theorem is simple when $m = 1$ or $n = 1$, so in the following we are going to assume $m, n \geq 2$.

(a) Show that there is $v_1 \in \mathbb{C}^n$ with $\|v_1\|_2 = 1$ such that $\|Av_1\|_2 = \|A\|_2 =: \sigma_1$.

(b) Show that for unitary matrix $U \in \mathbb{C}^{m\times m}$, $\|UA\|_2 = \|A\|_2$.

(c) Define $u_1 = Av_1/\|Av_1\| \in \mathbb{C}^m$, then $Av_1 = \sigma_1 u_1$. Extend u_1 to an o.n. basis $\{u_1, \ldots, u_m\}$ of $\mathbb{C}^m$ and v_1 to an o.n. basis $\{v_1, \ldots, v_n\}$ of $\mathbb{C}^n$. Let U_1 be the matrix with columns u_i and V_1 be that with columns v_i, then

$$U_1^* A V_1 = [A]^{(u_1,\ldots,u_m)}_{(v_1,\ldots,v_n)} = \begin{array}{c} \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{array} \begin{array}{c} \begin{array}{cccc} Av_1 & Av_2 & \cdots & Av_n \end{array} \\ \left[\begin{array}{c|c} \sigma_1 & w^*(\text{to be proved } \mathcal{O}) \\ \hline \mathcal{O} & B := [A]^{(u_2,\ldots,u_m)}_{(v_2,\ldots,v_n)} \end{array}\right] \end{array} =: S.$$

Show that $w = 0 \in \mathbb{C}^{n-1}$ by considering $\left\| S \begin{pmatrix} \sigma_1 \\ w \end{pmatrix} \right\| \geq \sigma_1^2 + w^* w$.

Remark. *Note that we have $x \perp v_1 \implies Ax \perp Av_1$, and the only assumption to derive this result is $\|Av_1\|_2 = \|A\|_2$, with $\|v_1\|_2 = 1$. We extract this as a technical corollary.*

Corollary. *Let $A \in \mathbb{C}^{m\times n}$, $v \in \mathbb{C}^n$ with $\|v\|_2 = 1$. Then if $\|Av\|_2 = \|A\|_2$,*

$$w \perp v \implies Aw \perp Av.$$

The same is true when $\mathbb{C}$ is replaced by $\mathbb{R}$.

(d) Explain why $B = [A]^{(u_2,\ldots,u_m)}_{(v_2,\ldots,v_n)}$.

(e) Prove the existence part of SVD by induction on k, where $m + n = k$.

(f) Note, however, that if the choices of $\{u_2, \ldots, u_m\}$ and $\{v_2, \ldots, v_n\}$ change, the "Σ" may also be changed. Show that the resulting Σ in the existence part of the SVD is independent of such choices. So the uniqueness part is completed.

We have thereby proved the existence of SVD for any complex matrix.

Problem 307. Show that part (i) of the theorem quoted in problem 306 is also true if $\mathbb{C}$ are changed to $\mathbb{R}$, i.e., $A \in \mathbb{R}^{m\times n}$, $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$.

Problem 308. Two matrices $A, B \in \mathbb{C}^{m\times m}$ are unitarily equivalent if $A = QBQ^*$, for some unitary $Q \in \mathbb{C}^{m\times m}$. Is it true or false that A and B are uniarily equivalent if and only if they have the same singular values (i.e., same Σ in their SVDs)?

Problem 309. Using the SVD, prove that any matrix in $\mathbb{C}^{m\times n}$ is the limit of a sequence of matrices of full rank. In other words, prove that the set of full-rank matrices is a dense subset of $\mathbb{C}^{m\times n}$.

Problem 310. By considering the SVD of $A \in \mathbb{C}^{m\times m}$, say, $A = U\Sigma V^*$. Find an eigenvalue decomposition of the $2m \times 2m$ hermitian matrix

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}.$$

Here to find an eigenvalue decomposition is the same as to find an invertible matrix X such that $X^{-1}AX$ is diagonal.

Definition. Let $P \in \mathbb{F}^{n\times n}$. We say that P is a **projector** iff $P^2 = P$. Note that for each $x \in \mathbb{F}^n$, $x = Px + (x - Px)$, $Px \in \operatorname{range} P$ while $x - Px \in \operatorname{null} P$. So whenever a matrix P is a projector, $\mathbb{F}^n = \operatorname{range} P + \operatorname{null} P$. We say that P is **orthogonal** iff $\operatorname{range} P$ and $\operatorname{null} P$ are orthogonal.

Problem 311. Show that a projector P is orthogonal if and only if $P = P^*$.

Problem 312. Let $P \in \mathbb{C}^{m\times m}$ be a nonzero projector. Show that $\|P\|_2 \geq 1$, with equality holds if and only if P is an orthogonal projector.

Problem 313. By considering QR factorization, show that if $A = [\ a_1 \mid \cdots \mid a_n\ ]$, where $a_i \in \mathbb{F}^n$ is column vector for each i, one has

$$|\det A| \leq \prod_{i=1}^n \|a_i\|_2.$$

1.13 Algebra

Problem 314. Show that any group G cannot be a union of two proper subgroups.

Problem 315. Count the elements contained in the following sets.

(a) $\{\sigma \in S_5 : |\sigma| = 2\}$. (b) $\{\sigma \in S_n : |\sigma| = 2\}$. (c) $\{\sigma \in S_n : |\sigma| = p\}, p \leq n$.

Problem 316. If H and K are normal subgroups of a group G with $HK = G$, prove that

$$G/(H \cap K) \simeq (G/H) \times (G/K).$$

Problem 317. Let G be a group of order $p^s m$, where p is prime, $p \nmid m$. Let $H \leq G$ with order p^s and $K \leq G$ with order p^t, $0 < t \leq s$ and $K \nsubseteq H$, show that $HK \nleq G$.

Problem 318. Let A, B be two finite subgroups of G. Show that $|AB| = \dfrac{|A||B|}{|A \cap B|}$.

Problem 319. Prove that every even permutation in S_n is a product of cycles of length 3.

Problem 320. Let G be a finite group. For a given $a \in G$, let

$$S_a = \{x \in G : xax^{-1} = a^2\}.$$

If a has order n, prove that $|S_a|$ is a multiple of n.

Problem 321. Let G be a group, if every element in G has order 2, show that G must be abelian.

Problem 322. Let N be a given positive integer with $N \geq 3$, and let $a_1, a_2, \ldots, a_m$ be the full list of integers such that $1 \leq a_i \leq N-1$ and a_i is relatively prime to N (so $m = \phi(N)$).

(a) Prove that $a_1 + a_2 + \cdots + a_m$ is a multiple of N.

(b) If k is relatively prime to m, prove that $a_1^k + a_2^k + \cdots + a_m^k$ is a multiple of N.

Problem 323. Let G be a finite group with $|G| = p^n$, where p is a prime. Suppose G acts on a finite set X. Let B be the set of fixed points, that is,

$$B = \{x \in X : gx = x \text{ for all } g \in G\}.$$

Prove that $|X| - |B|$ is a multiple of p.

Problem 324. Let $F = \{a_1, a_2, \ldots, a_n\}$ be a finite field with n elements. If $n > 3$, prove that

$$a_1^2 + a_2^2 + \cdots + a_n^2 = 0 \quad \text{and} \quad \sum_{1 \leq i < j \leq n} a_i a_j = 0.$$

Problem 325. Suppose R is a commutative ring with 1, let $a, b, x, y \in R$ satisfy $ax + by = 1$.

(a) Prove that there exist $c, d \in R$ such that $cx^{2004} + dy^{2004} = 1$.

(b) Let I be an ideal such that $x^{2003} \in I$ and $y^{2004} \in I$, prove that $I = R$.

Problem 326. Let F be a finite field and $|F| = n > 2$. Let $a_1, a_2, \ldots, a_n$ be the list of all elements in F.

(a) For a non-zero element $b \in F$, prove that the list $a_1 b, a_2 b, \ldots, a_n b$ is a permutation of $a_1, a_2, \ldots, a_n$.

(b) Prove that $a_1 + a_2 + \cdots + a_n = 0$.

(c) Prove that for an arbitrary positive integer k, $a_1^k + a_2^k + \cdots + a_n^k$ is either 0 or -1.

Problem 327. (a) Let H, with order 2, be a normal subgroup of G, prove that H is in the center of G.

(b) Let H be a normal subgroup of prime order p in a finite group G. Suppose that p is the smallest prime dividing $|G|$. Prove that H is in the center $Z(G)$.

Problem 328. (a) Show that if $G/Z(G)$ is cyclic, then G is abelian.

(b) Show that a nonabelian group G of order pq, where p, q are primes, has (only) a trivial center.

Problem 329. Let $\vec{a}, \vec{b}$ be two linearly independent vectors in $\mathbb{R}^2$. Prove that every other pair of vectors $\vec{a}', \vec{b}'$ such that

$$\mathbb{Z}\vec{a} + \mathbb{Z}\vec{b} = \mathbb{Z}\vec{a}' + \mathbb{Z}\vec{b}'$$

must be of the form $(\vec{a}, \vec{b}) = (\vec{a}', \vec{b}')P$, where P is a 2×2 integer matrix with determinant ± 1.

Problem 330. We say that A is dense in B if any element of B is either an element of A or a limit (or called accumulation) point of A.

(a) Prove that a subgroup Γ of $(\mathbb{R}, +)$ is either dense in $\mathbb{R}$ or else of the form $\mathbb{Z}a$, for some $a > 0$.

(b) Show that $\mathbb{Z} + \mathbb{Z}\sqrt{2}$ is dense in $\mathbb{R}$. Generalize the result to $\mathbb{Z}q + \mathbb{Z}r$ for rational q and irrational r.

(c) Let H be a subgroup of the group G of rotations in $\mathbb{R}^2$. Prove that H is either a cyclic subgroup of G or else dense in G.

Problem 331. Let F be a field and $F^\times = F \setminus \{0\}$ be the multiplicative group of F.

(a) Prove that every finite subgroup of $F^\times$ is cyclic.

(b) Prove that for each positive integer n, $F^\times$ contains at most one subgroup of order n.

1.14 Hint

- Problem 133: Suppose that $\sum_{n=1}^{\infty} a_n^2 = \infty$, consider the partition of $\mathbb{N}$ into consecutive finite segments $E_1, E_2, E_3, \dots$ such that $\sum_{n \in E_k} a_n^2 > 1$. Define a sequence $\{b_n\}$ by setting $b_n = c_k a_n$, for $n \in E_k$. Choose suitable c_k to get contradiction.

- Problem 144: Show that $\displaystyle\sum_{\ell=1}^{k} \sum_{r=1}^{\ell} a_{r,\ell} = \sum_{r=1}^{k} \sum_{\ell=r}^{k} a_{r,\ell}$.

- Problem 147: Count all possibilities of picking out m numbers with replacement from $\{1, 2, \dots, n\}$, moreover, these possibilities must contain two specified numbers (say $1, 2$), also note to get RHS, we need inclusion-exclusion.

Chapter 2

Solutions

Beforehand two symbols are introduced here for simplicity. $\sum\limits_{cyc}$, $\sum\limits_{\text{cyc}}$ or $\sum\limits_{\text{cyclic}}$ means to take the **cyclic summation** about known variables $\sum\limits_{\text{cyclic}} f(a,b,c) = f(a,b,c) + f(b,c,a) + f(c,a,b)$. A **symmetric sum** is denoted by $\sum\limits_{sym}$ or $\sum\limits_{\text{sym}}$, and it is seldom written as $\sum\limits_{\text{symmetric}}$. The symmetric sum takes over all permutations about a, b, c in $f(a,b,c)$, there are $3! = 6$ such terms, that is

$$\sum_{\text{sym}} f(a,b,c) = f(a,b,c) + f(a,c,b) + f(b,a,c) + f(b,c,a) + f(c,a,b) + f(c,b,a).$$

For example, $(x+y+z)^3 = \sum\limits_{\text{cyclic}} x^3 + 3\sum\limits_{\text{sym}} x^2y + 6xyz = \sum\limits_{\text{cyclic}} (x^3 + 3x^2(y+z) + 2xyz)$.

2.1 Inequality

1. (a) Direct consequence of Cauchy-Schwarz inequality, equality holds if and only if their corresponding ratios are equal, i.e. $\frac{a}{c} = \frac{b}{d}$ or $ad = bc$.

(b) Making a transform before using the result, we have

$$\begin{aligned} f(x) &= 7 - 2x + \tfrac{1}{2}\sqrt{3 + 2x - x^2} \\ &= 5 + 2(1-x) + \tfrac{1}{2}\sqrt{3 + 2x - x^2} \\ &\le 5 + \sqrt{(4 + 1/2^2)(1 - 2x + x^2 + 3 + 2x - x^2)} \\ &= 5 + \sqrt{17}. \end{aligned}$$

Equality holds if and only if $2\sqrt{3 + 2x - x^2} = \frac{1}{2}(1-x)$ or $x = 1 - \dfrac{8}{\sqrt{17}}$.

2. $\sum\limits_{cyc} a(b+c) \sum\limits_{cyc} \dfrac{a}{b+c} \ge (a+b+c+d)^2$, now observe that

$$\begin{aligned} \sum_{cyc} a(b+c) &= ab + ac + ad + bc + bd + cd + ac + bd \\ &\le ab + ac + ad + bc + bd + cd + \frac{a^2 + c^2 + b^2 + d^2}{2} \\ &= \frac{(a+b+c+d)^2}{2}, \end{aligned}$$

so the result follows.

3. To be added.

4. Consider $(\sum_{cyc}\sqrt{a^2+3})^2 = (\sum_{cyc}\sqrt{a}\sqrt{a+3/a})^2 \le \sum_{cyc} a \sum_{cyc}(a+3/a) = 4(\sum_{cyc} a)^2$.

5. Summing up these two inequalities, we are done.

$$\sum_{cyc}(b+c)\sum_{cyc}\frac{b^2}{b+c} \ge (a+b+c)^2 \quad \text{and} \quad \sum_{cyc}(b+c)\sum_{cyc}\frac{c^2}{b+c} \ge (a+b+c)^2.$$

6. Cauchy

$$(b+c+c+a+a+b)\left(\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\right) \ge (a+b+c)^2$$
$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \ge \frac{a+b+c}{2} \ge \frac{3}{2}.$$

Another method, we try to show that $\sum_{\text{cyclic}}\frac{a^2}{b+c} \ge \frac{a+b+c}{2}$ first, since two sides of the inequality are homogeneous, we can normalise (which means we "set" a condition that is more favourable) to $a+b+c=1$ such that

$$\sum_{\text{cyclic}}\frac{a^2}{b+c} \ge \frac{a+b+c}{2} \quad \iff \quad \sum_{\text{cyclic}}\frac{a^2}{1-a} \ge \frac{1}{2}.$$

This can be easily proved by Cauchy-Schwarz inequality, another way to show is to let $f(x) = \frac{x^2}{1-x} \Longrightarrow f''(x) = \frac{2}{(1-x)^3}$, it shows that f is strictly convex on $(0,1)$ and hence

$$\begin{aligned} f(x) = \frac{x^2}{1-x} &\ge f'\left(\frac{1}{3}\right)\left(x-\frac{1}{3}\right)+f\left(\frac{1}{3}\right) \\ &= \frac{5}{4}\left(x-\frac{1}{3}\right)+\frac{1}{6} \end{aligned}$$

where right hand side is the tangent on f at $(\frac{1}{3}, f(\frac{1}{3}))$. Thus

$$\sum_{\text{cyclic}}\frac{a}{1-a} \ge \frac{5}{4}(a+b+c-1)+\frac{3}{6} = \frac{1}{2}.$$

7. Since $(3)(a^2+b^2+c^2) \ge (a+b+c)^2 = 1$, it follows that

$$\begin{aligned} (\sqrt{1-3a^2}+\sqrt{1-3b^2}+\sqrt{1-3c^2})^2 &\le (3)(3-3(a^2+b^2+c^2)) \\ &\le 6. \end{aligned}$$

8. By Cauchy-Schwarz inequality

$$\sum_{\text{cyclic}} a(b+c)\sum_{\text{cyclic}}\frac{1}{a^3(b+c)} \ge \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$$

we have[1]

$$\sum_{\text{cyclic}}\frac{1}{a^3(b+c)} \ge \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{2} \ge \frac{3}{2}\sqrt[3]{\frac{1}{abc}} = \frac{3}{2}.$$

[1] $\sum_{\text{cyclic}} a(b+c) = 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

Or, you may find this questions is similar to Q2. Replacing a, b, c respectively by $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, you find they are exactly the same!

Or, by AM-GM inequality, we have

$$\frac{a^2}{b} \geq \frac{2a}{k} - \frac{b}{k^2}$$

where a, b, k are positive reals. Now

$$\sum_{\text{cyclic}} \frac{1}{a^3(b+c)} = \sum_{\text{cyclic}} \frac{(\frac{1}{a})^2}{\frac{1}{c}+\frac{1}{b}} \geq \sum_{\text{cyclic}} \left(\frac{2(\frac{1}{a})}{k} - \frac{\frac{1}{c}+\frac{1}{b}}{k^2} \right).$$

We take $k = 2$ such that $1/k - 1/k^2 = 1/4$

$$\sum_{\text{cyclic}} \frac{1}{a^3(b+c)} \geq 2 \sum_{\text{cyclic}} \frac{1}{a} \left(\frac{1}{k} - \frac{1}{k^2} \right) \geq 6 \left(\frac{1}{k} - \frac{1}{k^2} \right) = \frac{3}{2}.$$

9. We eliminate 2, 3 and 6 in the expression $2b^2 + 3c^2 + 6d^2$, that gives

$$\begin{aligned} (b+c+d)^2 &\leq \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6} \right) (2b^2 + 3c^2 + 6d^2) \\ (3-a)^2 &\leq 5 - a^2 \\ (a-1)(a-2) &\leq 0 \\ 1 \leq a &\leq 2. \end{aligned}$$

10. We use Cauchy-Schwarz inequality twice, having

$$\left(\sum_{cyc} a \right) \left(\sum_{cyc} \frac{a^3}{b^2} \right) \geq \left(\sum_{cyc} \frac{a^2}{b} \right)^2 \quad \text{and} \quad \left(\sum_{cyc} b \right) \left(\sum_{cyc} \frac{a^2}{b} \right) \geq \left(\sum_{cyc} a \right)^2,$$

combining these two, we are done.

11. Refer to question 5, note that when taking square root on both sides, we should add a "±" sign as we concern all reals, in such way we can find maxima and minima.

12. Cauchy-Schwarz inequality applies to a set of reals not all zero, this fact enables us to cancel out terms by multiplying a negative factor.

$$(a^2 + b^2 + (2a - 3b + 4)^2)((-2)^2 + 3^2 + 1) \geq (-2a + 3b + 2a - 3b + 4)^2 = 16$$

$$a^2 + b^2 + (2a - 3b + 4)^2 \geq \frac{16}{14} = \frac{8}{7}.$$

13. (a) We apply Cauchy-Schwarz ineqaulity to eliminate the undesired factors.

$$\sum_{i=1}^{n} a_i(S - a_i) \sum_{i=1}^{n} \frac{a_i}{S - a_i} \geq \left(\sum_{i=1}^{n} a_i \right)^2 = S^2$$

while since

$$\sum_{i=1}^{n} a_i^2 \geq \frac{S^2}{n} \tag{2.1}$$

it follows that

$$\left(\frac{n-1}{n} \right) S^2 \sum_{i=1}^{n} \frac{a_i}{S - a_i} \geq S^2 \implies \sum_{i=1}^{n} \frac{a_i}{S - a_i} \geq \frac{n}{n-1}.$$

(b) By (1)

$$\left(\frac{n-1}{n}\right) S^2 \sum_{i=1}^{n} \frac{S-a_i}{a_i} \geq \sum_{i=1}^{n} a_i(S-a_i) \sum_{i=1}^{n} \frac{S-a_i}{a_i} \geq \left(\sum_{i=1}^{n}(S-a_i)\right)^2,$$

and hence

$$\sum_{i=1}^{n} \frac{S-a_i}{a_i} \geq n(n-1).$$

14. By Cauchy-Schwarz inequality, we have

$$\sum_{k=1}^{n} a_k(a_k+1) \sum_{k=1}^{n} \frac{a_k}{a_k+1} \geq \left(\sum_{k=1}^{n} a_k\right)^2$$

while since

$$\sum_{k=1}^{n} a_k \left(\sum_{k=1}^{n} a_k + 1\right) - \sum_{k=1}^{n} a_k(a_k+1) = \left(\sum_{k=1}^{n} a_k\right)^2 - \sum_{k=1}^{n} a_k^2 > 0,$$

this implies

$$\sum_{k=1}^{n} a_k \left(\sum_{k=1}^{n} a_k + 1\right) > \sum_{k=1}^{n} a_k(a_k+1),$$

the result follows.

15. The denominator of each fraction is obviously positive, for example, $a^2 - bc + 1 = a^2 - bc + 3(ab+bc+ca)$, which has no negative term. Observe that $\sum_{cyc} \frac{a}{a^2-bc+1} \sum_{cyc}(a^2-bc+1)a \geq \left(\sum_{cyc} a\right)^2 = \sum_{cyc} a^2 + \frac{2}{3}$, we have

$$\sum_{cyc} \frac{a}{a^2-bc+1} \geq \frac{\sum_{cyc} a^2 + \frac{2}{3}}{a^3+b^3+c^3-3abc+\sum_{cyc} a} = \frac{\sum_{cyc} a^2 + \frac{2}{3}}{\sum_{cyc} a(\sum_{cyc} a^2 - \frac{1}{3} + 1)} = \frac{1}{a+b+c}.$$

16. Method 1. We observe that for all $x > 0$, the inequality $x^k \geq k(x-1)+1$ holds for $k \in \{1, 2, \ldots, n\}$. So

$$\sum_{k=1}^{n} \frac{(x_k)^k}{k} \geq \sum_{k=1}^{n} \frac{k(x_k-1)+1}{k} = \sum_{k=1}^{n} x_k + \sum_{k=1}^{n} \frac{1}{k} - n,$$

equality holds when $x_1 = x_2 = \cdots = x_n = 1$.

Finally by Cauchy-Schwarz inequality (or AM-GM as in method 2), $n\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} \frac{1}{x_k} \sum_{k=1}^{n} x_k \geq n^2 \implies \sum_{k=1}^{n} x_k \geq n$, so

$$\sum_{k=1}^{n} \frac{(x_k)^k}{k} \geq n + \sum_{k=1}^{n} \frac{1}{k} - n = \sum_{k=1}^{n} \frac{1}{k}.$$

Method 2. Weighted AM-GM inequality states that if $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = 1$, then for $a_i > 0$,

$$\sum_{i=1}^{n} \omega_i a_i \geq \prod_{i=1}^{n} a_i^{\omega_i},$$

hence

$$\sum_{k=1}^{n}\frac{(x_k)^k}{k}=\sum_{k=1}^{n}\frac{1}{k}\sum_{k=1}^{n}\frac{\frac{1}{k}}{\sum_{k=1}^{n}\frac{1}{k}}(x_k)^k\geq\sum_{k=1}^{n}\frac{1}{k}(x_1x_2\ldots x_n)^{\sum_{k=1}^{n}\frac{1}{k}}\geq\sum_{k=1}^{n}\frac{1}{k},$$

the last inequality follows from

$$\sqrt[n]{a_1a_2\ldots a_n}\geq\frac{n}{1+\frac{1}{2}+\cdots+\frac{1}{n}}=1.$$

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Lagrange multipliers. Define $L(x_1,\ldots,x_n)=\sum_{k=1}^{n}\frac{x_k^k}{k}+\lambda\left(\sum_{k=1}^{n}\frac{1}{x_k}-n\right)$. Its partial derivative with respect to x_k is $x_k^{k-1}-\lambda\frac{1}{x_k^2}$, equaled to zero yields $x_k^{k+1}=\lambda$ for all k. But the polynomial $x^{k+1}-\lambda=0$ can only have one positive real root, so all x_k are equal, clearly to 1 - and this is the only critical point, which easily is seen to be a minimum. So the sought after minimum is $\sum_{k=1}^{n}\frac{1}{k}\approx\ln n$.

If x_k are allowed to be negative also, the result stays when k is even, but the discussion becomes complicated for k odd, since then we have to check all cases when the moduli $|x_k|$ are equal.

17. We observe that

$$\left(\sum_{cyc}\frac{ab}{c(c+a)}\right)=\sqrt{\sum_{cyc}\frac{ca}{b(b+c)}\sum_{cyc}\frac{ab}{c(c+a)}}\geq\sum_{cyc}\sqrt{\frac{c+a}{c+b}}\left(\frac{a}{c+a}\right),\tag{1}$$

now

$$\left(\sum_{cyc}\frac{a}{c+a}\right)^2\leq\sum_{cyc}\sqrt{\frac{c+a}{c+b}}\left(\frac{a}{c+a}\right)\sum_{cyc}\sqrt{\frac{c+b}{c+a}}\left(\frac{a}{c+a}\right),\tag{2}$$

but if we consider the rightmost factor,

$$\left(\sum_{cyc}\sqrt{\frac{c+b}{c+a}}\left(\frac{a}{c+a}\right)\right)^2\leq\sum_{cyc}\frac{a(c+b)}{(c+a)^2}\sum_{cyc}\frac{a}{c+a},$$

it is enough to prove that $\sum_{cyc}\frac{a(c+b)}{(c+a)^2}\leq\sum_{cyc}\frac{a}{c+a}$, thereafter combining inequality (2) and then (1), we are done.

$$\sum_{cyc}\frac{a(c+b)}{(c+a)^2}\leq\sum_{cyc}\frac{a}{c+a}\iff\sum_{cyc}\frac{ab}{(c+a)^2}\leq\sum_{cyc}\frac{a^2}{(c+a)^2}.$$

However, $\sum_{cyc}\frac{ab}{(c+a)^2}\leq\frac{1}{2}\left(\sum_{cyc}\frac{a^2}{(c+a)^2}+\underbrace{\sum_{cyc}\frac{b^2}{(c+a)^2}}_{\text{call it } S}\right)\leq\sum_{cyc}\frac{a^2}{(c+a)^2}$. We cannot assume WLOG $a\geq b\geq c$ as the original inequality varies when either two are interchanged. But the inequality holds true because S itself is always a reverse sum, no matter $a\geq b\geq c$, $a\geq c\geq b$, whatever, so by rearrangement inequality, we have proved our claim.

18. By AM-GM inequality, one get $\sum_{cyc}\frac{a}{\sqrt{(a^2+b^2)(b^2+c^2)}} \geq 2\sum_{cyc}\frac{a}{\sum_{cyc}a^2+b^2}$, by Cauchy-Schwarz inequality,

$$\begin{aligned}2\sum_{cyc}\frac{a}{\sum_{cyc}a^2+b^2} &\geq 2\frac{(\sum_{cyc}a)^2}{\sum_{cyc}a\sum_{cyc}a^2+\sum_{cyc}ab^2} \geq 2\frac{(\sum_{cyc}a)^2}{\sum_{cyc}a\sum_{cyc}a+\sum_{cyc}ab}\\ &\geq 2\frac{(\sum_{cyc}a)^2}{(\sum_{cyc}a)^2+\frac{1}{3}(\sum_{cyc}a)^2}\\ &= \frac{3}{2}.\end{aligned}$$

19. Let a be the root of $P(z)$. If $|a| \leq 1$, we are done. Suppose that $|a| > 1$, then

$$\begin{aligned}|a^n| &= |a_{n-1}a^{n-1}+a_{n-2}a^{n-2}+\cdots+a_0|\\ &\leq |a_{n-1}||a|^{n-1}+|a_{n-2}||a|^{n-2}|\cdots+|a_0|\\ &\leq \sqrt{|a_{n-1}|^2+|a_{n-2}|^2+\cdots+|a_0|^2}\sqrt{(|a|^{n-1})^2+(|a|^{n-2})^2+\cdots+1}\\ &= \sqrt{|a_{n-1}|^2+|a_{n-2}|^2+\cdots+|a_0|^2}\sqrt{\frac{|a|^{2n}-1}{|a|^2-1}}\end{aligned}$$

squaring both sides, multiplying both sides by $\frac{|a|^2-1}{|a|^{2n}-1}$ and adding both sides by 1, we deduce that

$$|a|^2 < |a|^2+\frac{|a|^2-1}{|a|^{2n}-1} = \frac{|a|^{2n+2}-1}{|a|^{2n}-1} \leq 1+|a_{n-1}|^2+|a_{n-2}|^2+\cdots+|a_0|^2.$$

20. $a^{m+n}+b^{m+n}-a^mb^n-a^nb^m = (a^m-b^m)(a^n-b^n) \geq 0$.

21. Let $(x,y,z) = (1-a,1-b,1-c)$, then $x+y+z=1$, and the inequality to prove is equivalent to

$$\prod_{cyc}\frac{1-x}{x} = \sum_{cyc}\frac{1}{x}-1 \geq 8.$$

Remark. *This inequality is equivalent to* $\sum_{cyc}a^3+3abc \geq \sum_{sym}a^2b$.

22. Note that $(a+b)(b+c)(c+a)+abc = (ab+bc+ca)(a+b+c)$, it suffices to prove that $\frac{1}{8}(a+b)(b+c)(c+a) \geq abc$, which is obvious.

23. This inequality is trivially true if either one of a_i, b_j is zero. Suppose no one of them can be zero, then by dividing both sides $\sqrt[n]{b_1b_2\cdots b_n}$ and setting $x_i = \frac{a_i}{b_i}$, it suffices to show that

$$(\sqrt[n]{x_1x_2\cdots x_n}+1)^n \leq (1+x_1)(1+x_2)\cdots(1+x_n).$$

We expand RHS,

$$\begin{aligned}&(1+x_1)(1+x_2)\cdots(1+x_n) = 1+\sum_i x_i+\sum_{i<j}x_ix_j+\cdots+x_1x_2\cdots x_n\\ &\geq 1+\binom{n}{1}\sqrt[n]{x_1x_2\cdots x_n}+\binom{n}{2}(\sqrt[n]{x_1x_2\cdots x_n})^2+\cdots+(\sqrt[n]{x_1x_2\cdots x_n})^n \qquad \text{(why?)}\\ &= (1+\sqrt[n]{x_1x_2\cdots x_n})^n.\end{aligned}$$

24. By previous result, we have

$$\sum_{\text{cyclic}} \frac{1}{a^3+b^3+abc} \le \sum_{\text{cyclic}} \frac{1}{a^2b+b^2a+abc} = \frac{a+b+c}{abc(a+b+c)} = \frac{1}{abc}.$$

25. By previous result,

$$\sum_{\text{cyclic}} \frac{x^7+y^7}{x^2y+xy^2} \ge \sum_{\text{cyclic}} \frac{x^5y^2+y^5x^2}{x^3+y^3} = \sum_{\text{cyclic}} x^2y^2 \ge 3.$$

26. $\displaystyle\sum_{\text{cyclic}} \frac{x^2}{y^2+z^2+yz} \ge \frac{2}{3} \sum_{\text{cyclic}} \frac{x^2}{y^2+z^2} \ge 1$, what happens?[2]

27. Let $a = \cos^2\alpha, b = \cos^2\beta, c = \cos^2\gamma$, then $a+b+c = 1$

$$\cot^2\alpha + \cot^2\beta + \cot^2\gamma = \sum_{\text{cyclic}} \frac{a^2}{1-a^2}$$
$$= \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

we are done[3]. If you want to apply Cauchy-Schwarz inequality here, then consider two sets of numbers $\{b+c, c+a, a+b\}$ and $\left\{\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}\right\}$.

28.

$$\sum_{\text{cyclic}} \frac{1}{(x+y)(x^2+y^2)} = \sum_{\text{cyclic}} \frac{1}{x^3+xy^2+yx^2+y^3} \ge \frac{1}{2} \sum_{\text{cyclic}} \frac{1}{x^3+y^3}$$
$$\ge \frac{9}{4(x^3+y^3+z^3)}$$
$$\ge \frac{3}{4}.$$

29. The difficulty can be eased by homogenising two sides.

$$\begin{aligned} & p+q \le 2 \\ \iff\ & (p+q)^3 \le 4(p^3+q^3) \\ \iff\ & 3(p^3+q^3-p^2q-pq^2) \ge 0 \\ \iff\ & p^3+q^3 \ge p^2q+pq^2 \end{aligned}$$

the particular case of previous result 1, as each step is reversible, we are done.

30. Since $abc = 1$, let $a = \dfrac{x}{y}, b = \dfrac{y}{z}, c = \dfrac{z}{x}$, the original inequality becomes

$$(x+y-z)(y+z-x)(z+x-y) \le xyz.$$

However, it follows from the following inequality

$$\left(\prod_{\text{cyclic}} (x+y-z)\right)^2 = \prod_{\text{cyclic}} (x+y-z)(x-y+z)$$
$$= \prod_{\text{cyclic}} (x^2-(y-z)^2) \le (xyz)^2.$$

[2]Nesbitt's inequality!
[3]Nesbitt's inequality!!

31. Let $a = \tan^2\alpha, b = \tan^2\beta, c = \tan^2\gamma$, where α, β, γ are acute angle, then the given condition becomes $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, and hence

$$abc = \prod_{\text{cyclic}} \frac{1-\cos^2\alpha}{\cos^2\alpha} = \prod_{\text{cyclic}} \frac{\cos^2\beta + \cos^2\gamma}{\cos^2\alpha}.$$

It is equivalent to prove

$$\prod_{\text{cyclic}} \frac{y+z}{x} \geq 8$$

this follows from AM-GM inequality

$$(x+y)(y+z)(z+x) \geq 8\sqrt{xyyzzx} = 8xyz.$$

32. (a) Straightforward.

(b) Straightforward, otherwise by Cauchy-Schwarz inequality, we have

$$a+b = \left(\frac{1}{a} + \frac{1}{b}\right)(a+b) \geq 2^2 = 4.$$

(c) The statement is true of $n = 1$. Suppose the statements holds for $n = k$, i.e

$$a^k + b^k \leq (a+b)^k - 2^{2k} + 2^{k+1}.$$

Now for $n = k+1$,

$$\begin{aligned}
&(a+b)^{k+1} - a^{k+1} - b^{k+1} \\
&= (a+b)^{k+1} + (a+b)(a^{k-1} + b^{k-1}) - (a+b)(a^k + b^k) \\
&\geq (a+b)(a^{k-1} + b^{k-1} + 2^{2k} - 2^{k+1}) && \text{(induction assumption)} \\
&\geq (a+b)\left(2\left(\frac{a+b}{2}\right)^{k-1} + 2^{2k} - 2^{k+1}\right) && \text{(convexity of function } y = x^n\text{)} \\
&\geq 2^2(2^k + 2^{2k} - 2^{k+1}) && \text{(result of part b)} \\
&= 2^{2(k+1)} + 2^{k+2}.
\end{aligned}$$

33. Since the inequality is homogeneous on both sides, we are free to set $abc = 1, a+b+c = 1, ab+bc+ca = 1$, whatever. Here we set $abc = 1$ such that the inequality becomes

$$\sqrt{a} + \sqrt{b} + \sqrt{c} + (a+b+c)^2 \geq 4\sqrt{3}\sqrt{a+b+c}.$$

It can be shown by applying AM-GM inequality twice, that

$$\begin{aligned}
\sqrt{a} + \sqrt{b} + \sqrt{c} + (a+b+c)^2 &= 12 \cdot \frac{\sqrt{a} + \sqrt{b} + \sqrt{c} + 9\left(\frac{a+b+c}{3}\right)^2}{12} \\
&\geq 12\left(\frac{a+b+c}{3}\right)^{3/2} \\
&= \frac{4\sqrt{3}(a+b+c)\sqrt{a+b+c}}{3} \\
&\geq 4\sqrt{3}\sqrt{a+b+c}.
\end{aligned}$$

34. It is trivially true when $n = 1$, suppose the proposition holds for $n = 1, 2, \ldots, k$, i.e.

$$\begin{aligned} x_1 &\le x_1; \\ x_2 &\le x_1 + \frac{x_2}{2}; \\ x_3 &\le x_1 + \frac{x_2}{2} + \frac{x_3}{3}; \\ &\vdots \\ x_k &\le x_1 + \frac{x_2}{2} + \cdots + \frac{x_k}{k}. \end{aligned}$$

We sum up all assumptions, this gives

$$\sum_{i=1}^{k}(k-i+1)\frac{x_i}{i} \ge \sum_{i=1}^{k} x_i$$

and hence

$$\sum_{i=1}^{k}\frac{x_i}{i} \ge \frac{(x_1+x_k)+(x_2+x_{k-1})+\cdots+(x_k+x_1)}{k+1} \ge \frac{kx_{k+1}}{k+1}$$

or

$$\sum_{i=1}^{k+1}\frac{x_i}{i} \ge x_{k+1}$$

thus the proposition is true of $n = k+1$.

35. Method 1. We let $p_i = \frac{a_i}{b_i}$, where a_i, b_i are integers with $a_i < b_i$, we also let $A_i = \prod_{k=1,k\ne i}^{n} b_k$ and $S = \prod_{i=1}^{n} b_i$, then

$$\sum_{i=1}^{n}\frac{a_i}{b_i} = 1 \iff a_1A_1 + a_2A_2 + \cdots + a_nA_n = S.$$

It follows that

$$\begin{aligned} \sum_{i=1}^{n} p_i x_i &= \frac{a_1A_1x_1 + a_2A_2x_2 + \cdots + a_nA_nx_n}{a_1A_1 + a_2A_2 + \cdots + a_nA_n} \\ &\ge \prod_{i=1}^{n} x_i^{a_iA_i/S} = \prod_{i=1}^{n} x_i^{p_i}. \end{aligned}$$

Method 2. Induction on n is more simple, suppose the statement holds for $n = k$, now for $n = k+1$, we have

$$\begin{aligned} \sum_{i=1}^{k+1} p_i x_i &= (1-p_{k+1})\sum_{i=1}^{k}\left(\frac{p_i}{1-p_{k+1}}\right)x_i + p_{k+1}x_{k+1} \\ &\ge (1-p_{k+1})\left(\prod_{i=1}^{k} x_i^{p_i/(1-p_{k+1})}\right) + p_{k+1}x_{k+1} \\ &\ge \prod_{i=1}^{k+1} x_i^{p_i}, \end{aligned}$$

here the second row has used the result of $n = 2$ (which is not yet proved in our proof).

Remark. *It is one of the simple ways to prove AM-GM inequality.*

What's more, this inequality is just a particular case ($\varphi(x) = -\ln x$) of the following inequality.

Theorem (Jensen's Inequality)**.** *Suppose $x_1, x_2, \dots, x_n \in I$, φ is convex on I, $\lambda_i > 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, then*

$$\sum_{i=1}^{n} \lambda_i \varphi(x_i) \geq \varphi\left(\sum_{i=1}^{n} \lambda_i x_i\right).$$

36. Cauchy-Schwarz inequality yields the following,

$$(by + cz)(bz + cy) \leq (b^2 + c^2)(y^2 + z^2),$$

so we have a neat expression

$$\sum_{cyc} \frac{a^2x^2}{(by + cz)(bz + cy)} \geq \sum_{cyc} \left(\frac{a^2}{b^2 + c^2}\right)\left(\frac{x^2}{y^2 + z^2}\right). \qquad (*)$$

Now we are given that $a \geq b \geq c$, it follows that

$$a^2 \geq b^2 \geq c^2 \quad \text{and} \quad \frac{1}{b^2 + c^2} \geq \frac{1}{a^2 + c^2} \geq \frac{1}{a^2 + b^2} \implies \frac{a^2}{b^2 + c^2} \geq \frac{b^2}{a^2 + c^2} \geq \frac{c^2}{a^2 + b^2},$$

similarly, $\dfrac{x^2}{y^2 + z^2} \geq \dfrac{y^2}{x^2 + z^2} \geq \dfrac{z^2}{x^2 + y^2}$, so by Chebychef's inequality, from (*)

$$\frac{\sum_{cyc}\left(\frac{a^2}{b^2+c^2}\right)\left(\frac{x^2}{y^2+z^2}\right)}{3} \geq \frac{\sum_{cyc} \frac{a^2}{b^2+c^2}}{3} \cdot \frac{\sum_{cyc} \frac{x^2}{y^2+z^2}}{3} \geq \frac{(\frac{3}{2})^2}{3^2} = \frac{1}{4},$$

thus we are done.

Remark. *The last inequality follows from a well-known inequality $\sum_{cyc} \frac{a}{b+c} \geq \frac{3}{2}$.*

37. We just prove the case when $x_i, y_i \geq 0$, once this case is proved, we can see that the inequality is also true for $x_i, y_i \leq 0$. By Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2$$

$$\left(\sum_{i=1}^{n} x_i y_i\right)^3 \leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 \left(\sum_{i=1}^{n} x_i y_i\right) \leq \left(\sum_{i=1}^{n} x_i^2\right)^{3/2} \left(\sum_{i=1}^{n} y_i^2\right)^{3/2}.$$

By Jensen's inequality, since $\phi(x) = x^{3/2}$ is convex, we have with $a_1 + a_2 + \cdots + a_n = 1$ and $a_i \geq 0$,

$$\phi\left(\sum_{i=1}^{n} a_i x_i\right) \leq \sum_{i=1}^{n} a_i \phi(x_i),$$

in particular, we take $a_i = \dfrac{1}{n}$, $i = 1, 2, \dots, n$, then

$$\left(\sum_{i=1}^{n} \frac{x_i}{n}\right)^{3/2} \leq \sum_{i=1}^{n} \frac{x_i^{3/2}}{n}.$$

It follows that

$$\left(\sum_{i=1}^n x_i^2\right)^{3/2}\left(\sum_{i=1}^n y_i^2\right)^{3/2} = n^3\left(\frac{\sum_{i=1}^n x_i^2}{n}\right)^{3/2}\left(\frac{\sum_{i=1}^n y_i^2}{n}\right)^{3/2}$$
$$\leq n^3\cdot\frac{\sum_{i=1}^n x_i^3}{n}\cdot\frac{\sum_{i=1}^n y_i^3}{n}$$
$$= n\sum_{i=1}^n x_i^3\sum_{i=1}^n y_i^3.$$

Alternatively,

$$\left(\sum_{i=1}^n x_iy_i\right)^4 \leq \left(\sum_{i=1}^n \sqrt{x_iy_i}\right)^2\left(\sum_{i=1}^n (x_iy_i)^{3/2}\right)^2 \leq \sum_{i=1}^n x_iy_i\left(\sum_{i=1}^n 1\right)\sum_{i=1}^n x_i^3\sum_{i=1}^n y_i^3.$$

38. $\displaystyle\sum_{cyc}\frac{1}{1+2b^2c}-1\geq 0 \iff \sum_{cyc}\frac{2(1-b^2c)}{3(1+2b^2c)}\geq 0$. Removing all denominator, we have

$$\sum_{cyc}(1-b^2c)(1+2c^2a)(1+2a^2b)\geq 0.$$

Making good use of the symbol of cyclic summation, we are able to expand it effectively. On direct expansion, we get

$$\frac{a^2b+b^2c+c^2a+1}{4}\geq (abc)^3.$$

We know that by AM-GM inequality, $\dfrac{a^2b+b^2c+c^2a+1}{4}\geq (abc)^{3/4}$. We also claim that

$$(abc)^{3/4}\geq (abc)^3,$$

but it is equivalent to

$$abc(1-(abc)^3)\geq 0 \iff abc\leq 1,$$

the last inequality follows from the identity $a+b+c=3$, so we are done.

39. By Cauchy-Schwarz inequality, we have

$$\frac{\sqrt{a^3+b^3}}{a^2+b^2}\cdot\frac{\sqrt{a+b}}{\sqrt{a+b}}\geq\frac{\sqrt{(a^2+b^2)^2}}{(a^2+b^2)\sqrt{a+b}}=\frac{1}{\sqrt{a+b}},$$

hence the original cyclic sum will have a neat lower bound,

$$\sum_{cyc}\frac{\sqrt{a^3+b^3}}{a^2+b^2}\geq\sum_{cyc}\frac{1}{\sqrt{a+b}},$$

now we try to expand it, while due to the homogeneity of the original inequality, we normalize to $ab+bc+ca=1$, thus it becomes

$$\sum_{cyc}\frac{1}{\sqrt{a+b}}=\frac{\sum_{cyc}\sqrt{b+c}\sqrt{c+a}}{\sqrt{\prod_{cyc}(a+b)}}=\frac{(a+b+c)\sum_{cyc}\sqrt{1+c^2}}{(a+b+c)\sqrt{\prod_{cyc}(a+b)}}.$$

From the direct expansion and AM-GM inequality, we have $(a+b+c)^2\geq 3(ab+bc+ca)$, thus we have $a+b+c\geq\sqrt{3}$. Moreover, we observe that

$$\sqrt{1+x^2}\geq\tfrac{1}{2}(x-1/\sqrt{3})+2/\sqrt{3},$$

thus

$$\sum_{cyc} \sqrt{1+c^2} \geq \frac{1}{2}\left(a+b+c-\frac{3}{\sqrt{3}}\right)+\frac{6}{\sqrt{3}} \geq \frac{1}{2}(\sqrt{3}-\sqrt{3})+6/\sqrt{3}=\frac{6}{\sqrt{3}},$$

it follows that

$$\sum_{cyc} \frac{\sqrt{a^3+b^3}}{a^2+b^2} \geq \frac{(a+b+c)\sum_{cyc}\sqrt{1+c^2}}{(a+b+c)\sqrt{\prod_{cyc}(a+b)}} \geq \frac{6}{(a+b+c)\sqrt{\prod_{cyc}(a+b)}},$$

we are done.

40. To be added.

41. To be added.

42. Let $\lambda_i = \dfrac{b_i^3}{\sum_{i=1}^n b_i^3}$ and let $x_i = \dfrac{a_i}{b_i}$ in the inequality $\left(\sum_{i=1}^n \lambda_i x_i\right)^3 \leq \sum_{i=1}^n \lambda_i x_i^3$.

43. Observe that $\sum_{cyc} \dfrac{x-1}{x} = 1$, we have

$$x+y+z=\left(\sum_{cyc} x\right)\left(\sum_{cyc}\frac{x-1}{x}\right) \geq \left(\sum_{cyc}\sqrt{x-1}\right)^2 \implies \sqrt{x+y+z} \geq \sum_{cyc}\sqrt{x-1}.$$

44. We see that $4+9x^2=(3x+2+2\sqrt{3x})(3x+2-2\sqrt{3x})$. Cauchy-Schwarz inequality gives us

$$\begin{aligned}\sum_{cyc}\sqrt{4+9x^2} &\leq \sqrt{\left(\sum_{cyc}(3x+2+2\sqrt{3x})\right)\left(\sum_{cyc}(3x+2-2\sqrt{3x})\right)}\\ &= \sqrt{\left(3\sum_{cyc}x+6\right)^2-\left(2\sqrt{3}\sum_{cyc}\sqrt{x}\right)^2}\\ &= \sqrt{9\left(\sum_{cyc}x\right)^2+12\cdot 3+24\sum_{cyc}x-24\sum_{cyc}\sqrt{xy}}.\end{aligned}$$

By the fact that $xyz=1$, we have $\sum_{cyc}\sqrt{xy} \geq 3$, we also let $u=x+y+z$, and hence

$$\begin{aligned}&\sqrt{9\left(\sum_{cyc}x\right)^2+12\cdot 3+24\sum_{cyc}x-24\sum_{cyc}\sqrt{xy}}\\ \leq\ &\sqrt{9u^2+12\sum_{cyc}\sqrt{xy}+24u-24\sum_{cyc}\sqrt{xy}}\\ =\ &\sqrt{9u^2+24u-12\sum_{cyc}\sqrt{xy}}\\ \leq\ &\sqrt{9u^2+24u-36}.\end{aligned}$$

It suffices to show that $9u^2+24u-36 \leq 13u^2$, this inequality is equivalent to $(u-3)^2 \geq 0$, this is indeed true for any u, and hence we are done.

45. The original inequality is equivalent to $\sum_{cyc} x^3y^3 + \sum_{cyc} x^2y^4 \geq \sum_{cyc} x^3yz^2 + \sum_{cyc} x^2y^2z^2$. By Muirhead Inequality, $3 > 2, 3+3 > 2+2, 3+3+0 = 2+2+2$, yielding.

$$\sum_{sym} x^3y^3 \geq \sum_{sym} x^2y^2z^2 \implies \sum_{cyc} x^3y^3 \geq \sum_{cyc} x^2y^2z^2.$$

We are left to show that $\sum_{cyc} x^2y^4 \geq \sum_{cyc} x^3yz^2 \iff \sum_{cyc} \frac{x^2}{y^2} \geq \sum_{cyc} \frac{x}{y}$. By AM-GM inequality, we have $\frac{1}{3}\sum_{cyc} \frac{x}{y} \geq 1$, while by Cauchy-Schwarz inequality, we have

$$\sum_{cyc} \frac{x^2}{y^2} \geq \frac{1}{3}\left(\sum_{cyc} \frac{x}{y}\right)^2 \geq \sum_{cyc} \frac{x}{y}.$$

46. Observe that $\frac{a^2(b+1)}{a+b+ab} = a - \frac{ab}{a+b+ab}$, thus the inequality is equivalent to $\sum_{cyc} \frac{ab}{a+b+ab} \leq 1$. By AM-HM inequality, we have

$$\sum_{cyc} \frac{ab}{a+b+ab} = \frac{1}{3}\sum_{cyc} \frac{3}{1+\frac{1}{a}+\frac{1}{b}} \leq \frac{1}{9}\sum_{cyc}(1+a+b) = 1.$$

47. By Cauchy-Schwarz, we have

$$\sum_{cyc} \sqrt{\frac{a^2+b^2}{2c+a+b-2}} \geq \frac{1}{\sqrt{2}}\sum_{cyc} \frac{a+b}{\sqrt{2c+a+b-2}}.$$

Method 1. For every $s > 2$, observe that

$$\frac{s-x}{\sqrt{s+x-2}} \geq \frac{-3x+1+2s}{2\sqrt{s-1}},$$

this inequality follows from the convexity of $f(x) = \frac{s-x}{\sqrt{s+x-2}}$, and the fact that $f(x) \geq f'(1)(x-1) + f(1)$, letting $s = x+y+z$ and summing them up, we have

$$\frac{1}{\sqrt{2}}\sum_{cyc} \frac{s-a}{\sqrt{s+a-2}} \geq \frac{1}{\sqrt{2}} \cdot \frac{3(s+1)}{2\sqrt{s-1}} = \frac{3}{\sqrt{2}} \cdot \frac{\frac{s-1+2}{2}}{\sqrt{s-1}} \geq 3,$$

the last inequality follows from AM-GM inequality.

Method 2. Let $s = a+b+c$, then $\frac{1}{\sqrt{2}}\sum_{cyc} \frac{a+b}{\sqrt{2c+a+b-2}} = \frac{1}{\sqrt{2}}\sum_{cyc} \frac{s-c}{\sqrt{s-2+c}}$. We see that $\sum_{cyc} \sqrt{s-2+c}(s-c) \sum_{cyc} \frac{s-c}{\sqrt{s-2+c}} \geq 4s^2$, on the other hand,

$$\begin{aligned}
\left(\sum_{cyc} \sqrt{s-2+c}(s-c)\right)^2 &\leq \sum_{cyc}(s-c)(s-2+c)\sum_{cyc}(s-c) \\
&= \left(3s^2 - 6s + 2s - \sum_{cyc} a^2\right)(2s) \\
&\leq \left(3s^2 - 4s - \frac{1}{3}s^2\right)(2s) \\
&= \frac{8}{3}s(2s^2 - 3s).
\end{aligned}$$

As a result, $\frac{1}{\sqrt{2}}\sum_{cyc}\frac{s-c}{\sqrt{s-2+c}} \geq \frac{\sqrt{3}s}{\sqrt{2s-3}} = \frac{\sqrt{3}\cdot\frac{2s-3+3}{2}}{\sqrt{2s-3}} \geq 3.$

48. We first let $(a,b,c)=(a^3,b^3,c^3)$. By direct expansion and cancel out all denominator, we have

$$\sum_{sym} a^{12}b^3c^3 + \sum_{sym} a^9b^3c^3 + \sum_{sym} a^6b^3c^3 \leq \sum_{sym} a^{14}b^2c^2 + \sum_{sym} a^{13}bc + \sum_{sym} a^{12}.$$

Comparing the sums of the like power on both sides and applying Muirhead Inequality, we are done.

49. We divide both sides by $(k_1k_2\dots k_n)^m$, then

$$\left(\sum_{i=1}^{n}\frac{a_{1_i}^m}{k_1^m}\right)\left(\sum_{i=1}^{n}\frac{a_{2_i}^m}{k_2^m}\right)\cdots\left(\sum_{i=1}^{n}\frac{a_{m_i}^m}{k_m^m}\right) \geq \left(\sum_{i=1}^{n}\frac{a_{1_i}}{k_1}\frac{a_{2_i}}{k_2}\cdots\frac{a_{m_i}}{k_m}\right)^m.$$

We let $k_i = \sqrt[m]{a_{i_1}^m + a_{i_2}^m + \cdots + a_{i_n}^m}$ and let $x_{i_j} = \frac{a_{i_j}}{k_i}$ such that

$$x_{1_1}^m + x_{1_2}^m + \cdots + x_{1_n}^m = 1$$
$$x_{2_1}^m + x_{2_2}^m + \cdots + x_{2_n}^m = 1$$
$$\cdots$$
$$x_{m_1}^m + x_{m_2}^m + \cdots + x_{m_n}^m = 1,$$

and the inequality is equivalent to

$$\sum_{i=1}^{n} x_{1_i}x_{2_i}\cdots x_{m_i} \leq 1.$$

But from AM-GM inequality, we have

$$\sum_{i=1}^{n} x_{1_i}x_{2_i}\cdots x_{m_i} \leq \sum_{i=1}^{n}\frac{x_{1_i}^m + x_{2_i}^m + \cdots + x_{m_i}^m}{m} = \frac{\sum_{i=1}^n x_{1_i}^m + \sum_{i=1}^n x_{2_i}^m + \cdots + \sum_{i=1}^n x_{m_i}^m}{m} = 1,$$

we are done.

50. Observe that $\sum_{cyc} a^5x^6 = \sum_{cyc} a^{55/11}x^{66/11} \leq \sqrt[11]{\left(\sum_{cyc} a^{11}\right)^5\left(\sum_{cyc} x^{11}\right)^6} \leq 1.$

51. By the extended Cauchy-Schwarz inequality, we have

$$\sum_{cyc}\frac{a^3}{x}\sum_{cyc}x\sum_{cyc}1 \geq (a+b+c)^3 \iff \frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z} \geq \frac{(a+b+c)^3}{3(x+y+z)},$$

applying this inequality once, we have

$$\sum_{cyc}\frac{x^3}{1+9y^2xz} \geq \frac{(x+y+z)^3}{3\big(3+9xyz(x+y+z)\big)}.$$

It remains to show that

$$3+9xyz(x+y+z) \leq 6 \iff 3xyz(x+y+z) \leq 1 \iff 3xyz(x+y+z) \leq (xy+yz+zx)^2,$$

the last one holds as we know that $(a+b+c)^2 \geq 3(ab+bc+ca)$, hence we are done.

52. Although there is no any term with power 3 in numerator, extended Cauchy-Schwarz inequality also gives us a nice lower bound,

$$\sum_{cyc}\frac{a^2}{b(ma+nb)}\sum_{cyc}ab\sum_{cyc}(ma+nb)\ge(a+b+c)^3 \iff \sum_{cyc}\frac{a^2}{b(ma+nb)}\ge\frac{(a+b+c)^2}{(ab+bc+ca)(m+n)}.$$

It suffices to show that $(a+b+c)^2 \ge 3(ab+bc+ca)$, this is indeed true.

53. We use the usual substitution, $(a,b,c)=(x+y,x+z,y+z)$, $x,y,z>0$, then the original inequality is equivalent to $x^3y+y^3z+z^3x \ge xyz(x+y+z)$. Finally, the last inequality is easy enough to see by noting that $x^3y+y^3z+z^3x = xyz\left(\frac{x^2}{z}+\frac{y^2}{x}+\frac{z^2}{y}\right)$, we are done.

54. Note that $(a^3+1)(a^3+1)(b^3+1)\ge(a^2b+1)^3$, we are done (you can see what happens when $(a,b)=(b,c)$ and $(a,b)=(c,a)$).

55. We see that

$$a\cdot\frac{a+b}{2}\cdot\frac{a+b+c}{3}=\left(\frac{a}{3}+\frac{a}{3}+\frac{a}{3}\right)\left(\frac{a}{3}+\frac{a+b}{6}+\frac{b}{3}\right)\left(\frac{a}{3}+\frac{b}{3}+\frac{c}{3}\right)\ge\left(\frac{a}{3}+\sqrt[3]{\frac{ab(a+b)}{2\cdot 3^3}}+\frac{\sqrt[3]{abc}}{3}\right)^3.$$

Finally, $\frac{a+b}{2}\ge\sqrt{ab}$ does solve the problem.

56. Use $\sum_{cyc}\frac{a^3}{x}\ge\frac{(a+b+c)^3}{3(x+y+z)}$ once, we have a new lower bound $\sum_{cyc}\frac{a^6}{b^2+c^2}\ge\frac{1}{6}(a^2+b^2+c^2)^2$, since $(a^2+b^2+c^2)^2\ge(ab+bc+ca)^2\ge 3(abbc+bcca+caab)=3abc(a+b+c)$, we are done.

57. The left inequality is obviously true. To go through right inequality, we first determine the position of a,b,c,d and $c+d-b$. Observe that the inequality will change nothing if we interchange c and d, so without loss of generality, we assume that $c\le d$, finally $c+d-b\le c \iff d\le b$ we know that $a\le c+d-b\le c\le d\le b$.

Since $f(c)+f(d)\le f(c+d-b)+f(b) \iff \frac{f(b)-f(c)}{b-c}\ge\frac{f(d)-f(c+d-b)}{d-(c+d-b)}$, the equivalent inequality is obvious by convexity.

58. We observe the ineqality

$$\frac{\sin x}{x}\le\left(\frac{\frac{\pi}{3}\cdot\frac{1}{2}-\frac{\sqrt{3}}{2}}{\left(\frac{\pi}{3}\right)^2}\right)\left(x-\frac{\pi}{3}\right)+\frac{3\sqrt{3}}{2\pi}.$$

Well I havent rigorously check its validity (one variable case is simple, differentiation!), at least it is true for $x\in(0,\pi)$ with the aid of graph plotting, hence replacing x by respectively a,b,c and adding them up, we get desired result.

59. Define $h(x,y,z)=xy+yz+zx-2xyz=z(x+y)+(1-2z)xy$. Without loss of generality, assume $x\ge y\ge z$, since $1-2z=x+y+z-2z=x+y-z>0$, $h(x,y,z)\ge 0$. Now define $f(x,y)=h(x,y,z)$, we first keep z fixed and let x,y to be variable. In this way since $x+y+z=1$, $x+y$ is also fixed, but xy can still be varied. Let two constants $A=z(x+y), B=1-2z$, then

$$f(x,y)=A+Bxy\le A+B\left(\frac{x+y}{2}\right)^2.$$

The equality holds if and only if $x=y$, we now force x and y to be equal and move z, i.e. find the maximum of $F(x,y,z)=f(x,y)$. By $x+y+z=1$, we have $z=1-2x$, then $F(x,y,z)=g(x)=4x^3-5x^2+2x$. Now $g'=0 \implies x=\frac{1}{2}$ or $\frac{1}{3}$, $g''(\frac{1}{3})<0$, we have $g(x)\le g(\frac{1}{3})=\frac{7}{27}$.

60. $\sum_{cyc} \sqrt{a^2+(a-b)^2} = \sum_{cyc} a\sqrt{1+\left(\frac{1-b}{a}\right)^2} \geq \sum_{cyc} a\left(\frac{1}{\sqrt{2}}\left(\frac{1-b}{a}-1\right)+\sqrt{2}\right) = \frac{3\sqrt{2}}{2}$, here we have used an inequality $\sqrt{1+x^2} \geq \frac{1}{\sqrt{2}}(x-1)+\sqrt{2}$.

61. By rearrangement inequality, $\sum_{cyc} \frac{b+c}{\sqrt{a}} \geq \sqrt{a}+\sqrt{b}+\sqrt{c}+3\cdot\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{3} \geq \sqrt{a}+\sqrt{b}+\sqrt{c}+3$.

62. We use Cauchy-Schwarz inequality twice,

$$\begin{aligned}
\text{LHS} &\leq \sqrt{\sum_{cyc} a^2 \sum_{cyc} x^2} + \sqrt{\sum_{cyc} ab \sum_{cyc} xy} + \sqrt{\sum_{cyc} ab \sum_{cyc} xy} \\
&\leq \sqrt{\left(\sum_{cyc} a^2 + \sum_{cyc} ab + \sum_{cyc} ab\right)\left(\sum_{cyc} x^2 + \sum_{cyc} xy + \sum_{cyc} xy\right)} \\
&= \sqrt{(a+b+c)^2(x+y+z)^2} = a+b+c.
\end{aligned}$$

63. By Cauchy-Schwarz inequality, $\sum_{cyc} \frac{x}{xy+1} \geq \frac{9}{\sum_{cyc}(y+\frac{1}{x})} = \frac{9}{1+\frac{xy+yz+zx}{xyz}}$, it suffices to show that $\frac{9}{1+\frac{xy+yz+zx}{xyz}} \geq \frac{36xyz}{13xyz+1}$, this is equivalent to $9xyz+1 \geq 4\sum_{cyc} xy = 4\sum_{cyc} xy(x+y+z) = 4\sum_{sym} x^2y + 12xyz \iff 1 \geq 4\sum_{sym} x^2y + 3xyz$, finally by using the fact that $1=(x+y+z)^3$, the last inequality is equivalent to

$$\sum_{cyc} x^3 + 3xyz \geq \sum_{sym} x^2y,$$

this is true by Schur's inequality, $\sum_{cyc} x(x-y)(x-z) \geq 0$.

64. We denote $\sum = \sum_{i=1}^n$.

(a) By using Cauchy-Schwarz inequality once, we get

$$\left(\sum x_i^2\right)^2 = \left(\sum x_i^{1/2} x_i^{3/2}\right)^2 \leq \sum x_i \sum x_i^3. \quad (*)$$

Again, by Cauchy-Schwarz inequality,

$$\sum x_i \leq \sqrt{n}\sqrt{\sum x_i^2},$$

plugging in this into (*), we are done.

(b) We prove by inducting m. Suppose the statement holds when m is replaced by $m-1$, i.e.

$$\frac{\sum_{i=1}^n x_i}{n} \leq \left(\frac{\sum_{i=1}^n x_i^{m-1}}{n}\right)^{1/(m-1)}. \quad (**)$$

In exactly the same way, we have

$$\left(\sum x_i^{k-1}\right)^2 = \left(\sum x_i^{(k-2)/2} x_i^{k/2}\right)^2 \leq \sum x_i^{k-2} \sum x_i^k.$$

Recall that $x_i > 0, i = 1,2,\ldots,n$, we have

$$\frac{\sum x_i^{k-1}}{\sum x_i^k} \leq \frac{\sum x_i^{k-2}}{\sum x_i^{k-1}} \implies \prod_{k=2}^m \frac{\sum x_i^{k-1}}{\sum x_i^k} \leq \prod_{k=2}^m \frac{\sum x_i^{k-2}}{\sum x_i^{k-1}},$$

that implies $umx_i^{m-1} \leq \frac{numx_i^m}{umx_i}$, plugging in this into (**), we complete the induction.

(c) Replace β_i by $\frac{a_i}{b_i}$ and multiply both numerator and denominator a scale $L = \text{lcm}(b_1, b_2, \ldots, b_n)$ on both sides of inequality, we can note that every β_i is replaced by a $L(\frac{a_i}{b_i}) \in \mathbb{N}$. Hence without loss of generality, we can assume that $\beta_i \in \mathbb{N}$. Then by using part (b), we have

$$\frac{\sum_{i=1}^{n} \overbrace{x_i + \cdots + x_i}^{\beta_i' \text{s}}}{\beta} \le \left(\frac{\sum_{i=1}^{n} \overbrace{x_i^m + \cdots + x_i^m}^{\beta_i' \text{s}}}{\beta} \right)^{1/m}.$$

65. We see that the original inequality becomes

$$a^2b + b^2c + c^2a \ge a + b + c \iff (a^2b + b^2c + c^2a)^3 \ge (a + b + c)^3(abc)^2,$$

by simplifying this inequality, we get

$$\sum_{sym} a^6b^3 + 2\sum_{sym} a^5b^2c^2 + 3\sum_{sym} a^4b^4c \ge 6\sum_{sym} a^4b^3c^2.$$

66. First the original inequality is equivalent to

$$\sum_{cyc} x\sqrt{\frac{yz}{(x+y)(x+z)}} \le \frac{1}{2},$$

and we see that

$$\left(\sum_{cyc} x\sqrt{\frac{yz}{(x+y)(x+z)}}\right)^2 \le xyz\sum_{cyc} \frac{1}{(x+y)(x+z)},$$

finally we simplify it to

$$xyz\sum_{cyc} \frac{1}{(x+y)(x+z)} \le \frac{1}{4} \iff 8xyz \le (x+y)(y+z)(z+x).$$

67. We see that $\sum_{cyc} \frac{a+3}{3a+bc} \ge 3 \iff \sum_{cyc}(a+3)(3b+ca)(3c+ab) \ge 3\prod_{cyc}(3a+bc)$. We let $3u = a+b+c$, $3v^2 = ab+bc+ca$, $w^3 = abc$, then the inequality becomes

$$Aw^3 + B(u,v) \ge 3w^6 + C(u,v) \iff f(w^3) = -3w^6 + Aw^3 + B - C \ge 0,$$

here A is a constant, B and C are two functions independent of w. Since f is concave, it remains to prove that $f \ge 0$ when w^3 is maximal or minimal, both happen when either $a = b$ (two of them are equal) or $c = 0$ (one of them is zero), up to permutation.

In case if $a = b = x$, let $c = y \ne 0$ (if $a = b$, then we exclude the case $c = 0$ as they are mutually exclusive events), then the inequality becomes

$$2\frac{x+3}{3x+xy} + \frac{y+3}{3y+x^2} \ge 3 \iff \left(\frac{x}{y} - 1\right)^2 \ge 0.$$

In case if $c = 0$ (then $a + b = 3$), the inequality becomes $ab \le \frac{18}{7}$, this is true because $ab \le \left(\frac{3}{2}\right)^2 < \frac{18}{7}$.

68. Denote the area $A(x,\phi)$, one can show that the expression of A is

$$A = \frac{1}{2}\big((27 - 2x + 2x\cos\phi) + (27 - 2x)\big)x\sin\phi = \big(27x - (2 - \cos\phi)x^2\big)\sin\phi.$$

For fixed ϕ, A attains maximum when

$$x = \frac{27}{2(2-\cos\phi)}, \tag{*}$$

this results from completing square. It follows that $A \le \frac{27^2}{4}\left(\frac{\sin\phi}{2-\cos\phi}\right)$. One may intend to use differentiation to find the maximum of the latter factor, but we stick on "elementary" way. Observe that

$$\frac{\sin\phi}{2-\cos\phi} = \sqrt{\frac{1-\cos^2\phi}{(2-\cos\phi)^2}} \overset{u=2-\cos\phi}{=\!=\!=} \sqrt{\frac{1-(u-2)^2}{u^2}} = \sqrt{-1+4\left(\frac{1}{u}\right)-3\left(\frac{1}{u}\right)^2},$$

$\frac{\sin\phi}{2-\cos\phi}$ becomes a quadratic polynomial of $\frac{1}{u}$, by completing square again, maximum is attained when $\frac{1}{u}=\frac{2}{3}$, this means $u=\frac{3}{2}=2-\cos\phi \implies \phi = 60°$, plugging in this into (*), we get $x=9$.

69. By Hölder's inequality, we have $(a_i^3+1)(a_i^3+1)(a_{i+1}^3+1) \ge (a_i^2a_{i+1}+1)^3$. Define $a_{n+1}=a_1$, taking the product $\prod_{i=1}^n$ on both sides, we are done.

70. The inequality is equivalent to $\sum_{cyc}(1-a)\ln\frac{1}{a} \ge \ln 9$. As $\frac{d^2}{dx^2}\left((1-x)\ln\frac{1}{x}\right) = \frac{x+1}{x^2}$, we conclude that

$$\frac{\sum_{cyc}(1-a)\ln\frac{1}{a}}{3} \ge \left(1-\frac{a+b+c}{3}\right)\ln\left(\frac{1}{\frac{a+b+c}{3}}\right) \iff \sum_{cyc}(1-a)\ln\frac{1}{a} \ge \ln 9.$$

In general, for $a_1+a_2+\cdots+a_n=1$, $\sqrt{a_1^{1-a_1}a_2^{1-a_2}\cdots a_n^{1-a_n}} \le \frac{1}{n^{(n-1)/2}}$, replacing a_i by $\frac{a_i}{\sum_{i=1}^n a_i}$, we get desired inequality.

71. Method 1. We first make a usual substitution $(a,b,c)=(\cot\alpha,\cot\beta,\cot\gamma)$, where $\alpha,\beta,\gamma\in(0,\pi/2)$, then by the fact that $\tan^2 x+1=\sec^2 x$,

$$\prod_{cyc}\left(\frac{1}{a^2}+1\right) = 512 \iff \prod_{cyc}\cos\alpha = \frac{1}{2^{9/2}}.$$

The Cauchy-Schwarz inequality tells us

$$\sum_{cyc}\sin\alpha\times k = \sum_{cyc}\sin\alpha\sum_{cyc}\frac{\cos\alpha}{\sin\alpha} \ge \left(\sum_{cyc}\sqrt{\cos\alpha}\right)^2 \ge \frac{9\sqrt{2}}{4},$$

on the other hand,

$$\left(\sum_{cyc}\sin\alpha\right)^2 \le 3\sum_{cyc}\sin^2\alpha = 3\left(3-\sum_{cyc}\cos^2\alpha\right) \overset{\text{AM-GM}}{\le} 3\left(3-\frac{3}{8}\right) = \frac{63}{8} \implies \sum_{cyc}\sin\alpha \le \frac{\sqrt{126}}{4}.$$

Combing all results together, we get

$$\frac{\sqrt{126}}{4}\times k \ge \sum_{cyc}\sin\alpha\times k \ge \frac{9\sqrt{2}}{4} \implies k \ge \frac{9\sqrt{2}}{\sqrt{126}} = \frac{3}{\sqrt{7}},$$

the equality can hold, it is when $a=b=c=\frac{1}{\sqrt{7}}$ (not necessarily the only case).

Method 2.(From my net friend) Since

$$512 = \left(1+\frac{1}{a^2}\right)\left(1+\frac{1}{b^2}\right)\left(1+\frac{1}{c^2}\right) = 1+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+\left(\frac{1}{a^2b^2}+\frac{1}{b^2c^2}+\frac{1}{c^2a^2}\right)+\frac{1}{a^2b^2c^2}$$
$$\geq 1+\frac{3}{(abc)^{\frac{2}{3}}}+\frac{3}{(abc)^{\frac{4}{3}}}+\frac{1}{(abc)^2} = \left(1+\frac{1}{(abc)^{\frac{2}{3}}}\right)^3 \geq \left(1+\frac{1}{\left(\frac{k}{3}\right)^2}\right)^3,$$

it follows that

$$512 \geq \left(1+\frac{9}{k^2}\right)^3 \implies k \geq \frac{3\sqrt{7}}{7}.$$

72. Method 1. Observe that from hölder's inequality,

$$\left(\sum_{cyc}\frac{a^2}{(b+c)^2}\right)^2\sum_{cyc}\frac{a+b}{c} \geq \left(\sum_{cyc}\frac{a}{b+c}\right)^3.$$

Recall that $a^3+b^3 \geq ab(a+b)$. Now observe other two inequalities

$$2\frac{\sum_{cyc}a^3}{abc} \geq \sum_{cyc}\frac{a+b}{c} \quad \text{and} \quad \frac{1}{4}\frac{\sum_{cyc}a^3}{abc} \geq \sum_{cyc}\frac{a^2}{(b+c)^2}.$$

Combining all above, we are done.

Method 2. By direct expansion we know that the inequality is equivalent to

$$\sum_{sym}a^5b+\sum_{sym}a^4b^2 \geq \sum_{sym}a^3b^2c+6a^2b^2c^2,$$

so by Muirhead's inequality, we are done.

73. To be added.

74. It is enough to prove the case when $n=2$, we see that

$$\frac{x_1y_1}{x_1+y_1}+\frac{x_2y_2}{x_2+y_2}-\frac{(x_1+x_2)(y_1+y_2)}{x_1+x_2+y_1+y_2} \leq 0 \iff \frac{x_1^2y_2^2+x_2^2y_1^2}{2} \geq x_1x_2y_1y_2.$$

75. Direct consequence of Schur's inequality.

76. Use $\dfrac{2abc+1}{3} \geq (abc)^{2/3}$, then use Schur's inequality and muirhead inequality once.

77. Since both sides are symmetric polynomial of degree less than 5, by *uvw* method, it suffices to prove that the cases $a=b$ and $c=0$. If $a=b$, let $a=b=x$ and $c=y$, then the inequality becomes

$$x^2+b^2+2+x^2y+y \geq 2(x+y)+2xy,$$

this is obviously true. The case of $c=0$ is similar.

78. Since

$$\frac{xy}{x^2+y^2+2z^2} \leq \frac{1}{4}\frac{(x+y)^2}{x^2+y^2+2z^2} \leq \frac{1}{4}\left(\frac{x^2}{x^2+z^2}+\frac{y^2}{y^2+z^2}\right),$$

the last inequality follows from Cauchy-Schwarz of the form $\dfrac{(x+y)^2}{a+b} \leq \dfrac{x^2}{a}+\dfrac{y^2}{b}$. This is a useful upper bound, by general hölder we can show similar inequality of higher degree, so feel free to use AM-GM to get upper bound! We can always do something thereafter. Finally we note that

$$\frac{1}{4}\sum_{cyc}\left(\frac{x^2}{x^2+z^2}+\frac{y^2}{y^2+z^2}\right)=\frac{3}{4},$$

this is from direct expansion and the identity $(a+b)(b+c)(c+a)=\sum_{sym}a^2b+2abc$ (for simplicity we let $(a,b,c)=(x^2,y^2,z^2)$ first).

79. We see that $\sum_{cyc}\frac{1}{a+b^2+c^3}=\sum_{cyc}\frac{a+1+\frac{1}{c}}{(a+b^2+c^3)(a+1+\frac{1}{c})}\le\sum_{cyc}\frac{a+1+\frac{1}{c}}{(a+b+c)^2}$. Denote $S=a+b+c$, we know that $S\ge 3$, now

$$\sum_{cyc}\frac{1}{a+b^2+c^3}\le\frac{\sum_{cyc}(a+1+\frac{1}{c})}{S^2}=\frac{1}{S^2}(3+S+\sum_{cyc}ab)\le\frac{S+S+\frac{1}{3}S^2}{S^2}=\frac{1}{3}+\frac{2}{S}\le 1.$$

80. By the given inequality we know that $\frac{a}{B}\le\frac{a_i}{b_i}\le\frac{A}{b}$, so

$$\left(\frac{a_i}{b_i}-\frac{A}{b}\right)\left(\frac{a_i}{b_i}-\frac{a}{B}\right)\le 0,$$

on simplification, $a_i^2+\left(\frac{Aa}{Bb}\right)b_i^2\le\left(\frac{a}{B}+\frac{A}{b}\right)a_ib_i$. Having taken summation on both sides, we apply AM-GM inequality once, then

$$\sqrt{\sum_{i=1}^n a_i^2\sum_{i=1}^n b_i^2}\sqrt{\frac{Aa}{Bb}}\le\frac{1}{2}\left(\frac{a}{B}+\frac{A}{b}\right)\sum_{i=1}^n a_ib_i,$$

the result follows from some algebra.

81. Consider $1<2<\cdots<n-1$ and $1/n<1/(n-1)<\cdots<1/2$, and argue a little bit more.

82. Think about the tangent line $a^2\ge 6(a-3)+9$, or consider the inequality $a^2+a^2+\frac{27}{a}\ge 3a$.

83. By subtracting $a+b+c$ on both sides, the original inequality is equivalent to $\sum_{cyc}\frac{ab^2}{b^2+1}\le\frac{3}{2}$, this is easy to prove since LHS $\le\sum_{cyc}\frac{ab^2}{b}=\frac{1}{2}\sum_{cyc}ab$, finally the inequality $(\sum_{cyc}a)^2\ge 3\sum_{cyc}ab$ will do.

84. Observe that $\sum_{cyc}\sqrt{x}=\sum_{cyc}\sqrt{xy}\cdot\frac{1}{\sqrt{y}}$,

$$\sum_{cyc}\sqrt{x}\le\sqrt{\sum_{cyc}xy\sum_{cyc}\frac{1}{y}}=\sqrt{\frac{(\sum_{cyc}xy)^2}{xyz}}=\sqrt{\frac{(xyz-2)^2}{xyz}}.$$

Let $u=xyz$, it is enough to show that $\sqrt{\frac{(u-2)^2}{u}}\le\frac{3}{2}\sqrt{u}$, this is equivalent to

$$4(u-2)^2\le 9u^2\iff 5u^2+16u-16\ge 0\iff(u+4)(5u-4)\ge 0\iff u\ge\frac{4}{5},$$

recall that $u=xyz=xy+yz+zx+2>2>\frac{4}{5}$, we are done (in fact no equality).

2.2 Integration

85. (a) Let $I_n = \int_0^1 (1-x^{50})^n\, dx$, then integrate by parts, we have

$$I_n = (1-x^{50})^n \cdot x\Big|_0^1 - n\int_0^1 x(1-x^{50})^{n-1}(-50\cdot x^{49})\, dx = -50nI_n + 50nI_{n-1},$$

hence a recurrence relation arises,

$$\frac{I_n}{I_{n-1}} = \left(\frac{50n}{50n+1}\right).$$

Now substitute $n = 101$, the answer will be 5051.

(b) Let $I = \int_0^1 \left(\frac{(1-2x)e^x}{(e^x+e^{-x})^3} + \frac{(1+2x)e^{-x}}{(e^x+e^{-x})^3}\right) dx$, substitute $x = -u$ in the first integral, we have

$$I = \int_{-1}^1 \frac{(1+2x)e^{-x}\cdot e^{3x}}{(e^x+e^{-x})^3\cdot e^{3x}}\, dx = \int_{-1}^1 \frac{(1+2x)e^{2x}}{(e^{2x}+1)^3}\, dx = -\frac{1}{4}\int_{-1}^1 (1+2x)\, d(e^{2x}+1)^{-2}.$$

Integrate it by parts, we have $I = \frac{4-3e^2-e^{-2}}{4} + \frac{1}{2}\int_{-1}^1 \frac{de^x}{e^x(e^{2x}+1)^2}$, finally by the formula

$$\frac{1}{x(x^2+1)^2} = \frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2},$$

we have

$$I = \frac{e^2}{(e^2+1)^2}.$$

Having seen that the answer is such simple, we may guess alternative that is more simple would have eixsted.

(c)

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \frac{\sin nx}{(2^x+1)\sin x}\, dx &= \int_{\pi/2}^{-\pi/2} \frac{2^x(-\sin nx)}{2^x(2^{-x}+1)(-\sin x)}\,(-dx)\\ &= \int_{-\pi/2}^{\pi/2} \frac{\sin nx}{\sin x}\, dx - \int_{-\pi/2}^{\pi/2} \frac{\sin nx}{(2^x+1)\sin x}\, dx\end{aligned}$$

thus we have

$$I_n = \int_{-\pi/2}^{\pi/2} \frac{\sin nx}{(2^x+1)\sin x}\, dx = \int_0^{\pi/2} \frac{\sin nx}{\sin x}\, dx,$$

$$I_0 = 0 \quad\text{and}\quad I_1 = \int_0^{\pi/2} dx = \frac{\pi}{2}.$$

Carefully seeing that the pattern can be further simplified, we try to produce a term that can cancel out $\sin x$ in the denominator, for $n \ge 2$, we have

$$I_n - I_{n-2} = 2\int_0^{\pi/2} \frac{\cos(n-1)x\cdot\sin x}{\sin x}\, dx = \frac{2}{n-1}\sin\left((n-1)\frac{\pi}{2}\right) = -\frac{2}{n-1}\cos\left(\frac{n\pi}{2}\right).$$

When n is odd, we put $n = 2p-1$, where $p \ge 2$, and then

$$I_n = I_{2p-1} = I_{2p-3} = \cdots = I_1 = \frac{\pi}{2}.$$

When n is even, we put $n = 2p$, where $p \geq 1$ such that

$$I_{2p} - I_{2p-2} = \frac{2(-1)^{p+1}}{2p-1}$$
$$I_n = \sum_{p=1}^{\frac{n}{2}} (I_{2p} - I_{2p-2}) = \sum_{p=1}^{\frac{n}{2}} \frac{(-1)^{p+1}}{p - \frac{1}{2}}.$$

If n is negative, it can be seen that the integrand is an odd function, we are done.

(d) By the substitution $x = -u$, we get an equivalent integral, $I_n = \displaystyle\int_0^\pi \frac{\sin^2 nx}{\sin^2 x}\,dx$. Again, our aim is to get rid of the nuisance, $\sin x$, in the denominator, we do so as follows

$$J_n = I_n - I_{n-1} = \int_0^\pi \frac{\sin(2n-1)x}{\sin x}\,dx.$$

We repeat this process,

$$J_n - J_{n-1} = 2\int_0^\pi \cos(2n-2)x\,dx = 0,$$

and consequently $J_n = J_{n-1} = \cdots = \int_0^\pi dx = \pi$, thus $I_n = (n-1)\pi + I_1 = n\pi$.

(e) **Method 1.** Define

$$I = \int_0^\infty \frac{\tan^{-1}\pi x - \tan^{-1} x}{x}\,dx = \int_0^1 + \int_1^\infty = J_1 + J_2.$$

here the results follow from integration by parts

$$J_1 = -\lim_{x\to 0} \ln x(\tan^{-1}\pi x - \tan^{-1} x) - \int_0^1 \ln x\left(\frac{\pi}{1+(\pi x)^2} - \frac{1}{1+x^2}\right)dx,$$

and

$$J_2 = \lim_{x\to\infty} \ln x(\tan^{-1}\pi x - \tan^{-1} x) - \int_1^\infty \ln x\left(\frac{\pi}{1+(\pi x)^2} - \frac{1}{1+x^2}\right)dx.$$

Since from J_1,

$$\lim_{x\to 0} \ln x(\tan^{-1}\pi x - \tan^{-1} x) = \lim_{x\to 0} = \ln x(\pi - 1)x = 0,$$

and from J_2,

$$\lim_{x\to\infty} \ln x(\tan^{-1}\pi x - \tan^{-1} x) = \lim_{x\to\infty} \ln x\left(\cos^{-1}\frac{1}{\sqrt{(\pi x)^2+1}} - \cos^{-1}\frac{1}{\sqrt{x^2+1}}\right)$$
$$= \lim_{x\to\infty} \ln x\left(\frac{1}{x} - \frac{1}{\pi x}\right) = 0,$$

we have

$$I = J_1 + J_2 = \int_0^\infty \ln x\left(\frac{1}{1+x^2} - \frac{\pi}{1+(\pi x)^2}\right)dx,$$

it can be solved by substitution $x = \tan\theta$ and $x = \tan\theta/\pi$ respectively, yielding $I = \dfrac{\pi}{2}\ln\pi$.

This method is clumsy, however, calculation can be further simplified by the following

method.
Method 2.

$$\int_0^\infty \frac{\arctan \pi x - \arctan x}{x}\,dx = \int_0^\infty \frac{1}{x}\int_1^\pi d(\tan^{-1} yx)\,dx = \int_0^\infty \int_1^\pi \frac{1}{x}\cdot\frac{x}{1+(yx)^2}\,dy\,dx$$
$$= \int_1^\pi \frac{1}{y^2}\int_0^\infty \frac{1}{(1/y)^2+x^2}\,dx\,dy = \int_1^\pi \frac{1}{y^2}\left(\frac{1}{1/y}\cdot\tan^{-1}\frac{x}{1/y}\bigg|_0^\infty\right)dy$$
$$= \frac{\pi}{2}\int_1^\pi \frac{1}{y}\,dy$$
$$= \frac{\pi}{2}\ln\pi$$

(f)

$$\begin{aligned}\int_0^\infty \frac{1-\cos y}{ye^y}\,dy &= 2\int_0^\infty \frac{\sin^2(y/2)}{ye^y}\,dy \\ &= 2\int_0^\infty \frac{1}{ye^y}\int_0^1 d\left(\sin^2(ty/2)\right)\,dy \\ &= \int_0^1\int_0^\infty e^{-y}\sin(ty)\,dy\,dt.\end{aligned}$$

The inner integral can be computed by integration by parts twice, thus

$$\int_0^\infty \frac{1-\cos x}{xe^x}\,dx = \int_0^1 \frac{t}{t^2+1}\,dt = \frac{\ln 2}{2}.$$

(g) Similarly,

$$\begin{aligned}I &= \int_0^1 \frac{\tan^{-1} x}{x\sqrt{1-x^2}}\,dx = \int_0^1\int_0^1 d(\tan^{-1}(yx))\frac{dx}{x\sqrt{1-x^2}} \\ &= \int_0^1\int_0^1 \frac{1}{(1+(yx)^2)\sqrt{1-x^2}}\,dx\,dy.\end{aligned}$$

Substitute $x = \sin\theta$, we get an equivalent integral

$$I = 2\int_0^1\int_0^{\pi/2} \frac{1}{2+y^2-y^2\cos 2x}\,dx\,dy.$$

We next substitute $t(x) = \tan x$, note that t is bijective on the interval $[0,\pi/2)$, a unique pre-image is to be obtained in the computation $\left(\text{It is, for example, } \tan^{-1}(\infty) = \frac{\pi}{2}, \text{ but not } \frac{3\pi}{2}, \frac{5\pi}{2}, \dots\right)$. We get another form of the integral

$$I = \int_0^1\int_0^\infty \frac{1}{1+(y^2+1)x^2}\,dx\,dy = \int_0^1 \frac{1}{y^2+1}\int_0^\infty \frac{1}{(\sqrt{1/(y^2+1)})^2+x^2}\,dx\,dy.$$

From integration table, we get

$$I = \int_0^1 \frac{1}{\sqrt{y^2+1}}\left(\tan^{-1}\frac{x}{\sqrt{1/(y^2+1)}}\bigg|_0^\infty\right)dy = \frac{\pi}{2}\int_0^1 \frac{1}{\sqrt{x^2+1}}\,dx.$$

Finally the substitution $x = \tan\theta$ suffices to compute this integral

$$I = \frac{\pi}{2}\int_0^{\pi/4} \sec\theta\,d\theta = \frac{\pi}{2}\ln(\sqrt{2}+1).$$

(h) To be added.

(i) To be added.

(j) The substitution $x = \tan\theta$ can reduce the integral into one we are extremely familiar,

$$\int \frac{x^2-1}{(x^2+1)\sqrt{1+x^4}}\,\mathrm{d}x \overset{x=\tan\theta}{=\!=\!=} \int \frac{\tan^2\theta - 1}{\sqrt{1+\tan^4\theta}}\,d\theta = \int \frac{\sin^2\theta-\cos^2\theta}{\sqrt{\sin^4\theta+\cos^4\theta}}\,d\theta$$

$$= \int \frac{\frac{(1-\cos 2\theta)-(1+\cos 2\theta)}{2}}{\sqrt{(\sin^2\theta+\cos^2\theta)^2 - 2\left(\frac{\sin 2\theta}{2}\right)^2}}\,d\theta$$

$$= -\sqrt{2}\int \frac{\cos 2\theta}{\sqrt{2-\sin^2 2\theta}}\,d\theta = -\frac{1}{\sqrt{2}}\int \frac{d(\sin 2\theta)}{\sqrt{2-\sin^2 2\theta}}.$$

The formula of the type $\int \frac{dx}{\sqrt{a^2-x^2}}$ can be found in integration table, yielding

$$\int \frac{x^2-1}{(x^2+1)\sqrt{1+x^4}}\,dx = -\frac{1}{\sqrt{2}}\arcsin\left(\frac{\sin(2\arctan x)}{\sqrt{2}}\right) + C.$$

(k) To be added.

(l) Define $a_n = \int_0^\pi \frac{\cos nx - \cos na}{\cos x - \cos a}\,dx$. By writing $\cos nx = \cos[(n-1)x + x]$, one gets the following:

$$\begin{aligned}&\cos nx - \cos na\\ &= \cos((n-1)x)(\cos x - \cos a) + \cos a[\cos(n-1)x - \cos(n-1)a]\\ &\qquad\qquad\qquad - [\sin(n-1)x\sin x - \sin(n-1)a\sin a].\end{aligned}$$

Divide both sides by $\cos x - \cos a$ and integrate w.r.t x over $[0,\pi]$, the first term on RHS vanishes,

$$\begin{aligned} a_n &= \cos a \cdot a_{n-1} + \frac{1}{2}\left[(-2)\int_0^\pi \frac{\sin(n-1)x\sin x - \sin(n-1)a\sin a}{\cos x - \cos a}\,dx\right]\\ &= \cos a\cdot a_{n-1} + \frac{1}{2}(a_n - a_{n-2}),\end{aligned}$$

here we have used the observation that $a_n - a_{n-2} = (-2)\int_0^\pi \frac{\sin(n-1)x\sin x - \sin(n-1)a\sin a}{\cos x - \cos a}\,dx$, this easily follows from the identity $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$. We transpose terms to get $a_n - 2\cos a\cdot a_{n-1} + a_{n-2} = 0$. Fortunately this recurrence relation is nice enough. It is a routine calculation to solve it with $a_0 = 0$ and $a_1 = \pi$, so we get $a_n = \frac{\pi \sin na}{\sin a}$.

86. By AM-GM inequality, it is not hard to show

$$\prod_{i=1}^n \int_{\mathbb{R}} |f_i(x)|\,dx \geq \left(\int_{\mathbb{R}} |f_1(x)f_2(x)\cdots f_n(x)|^{1/n}dx\right)^n. \tag{2.2}$$

We prove it as follows: If one of $\int_{\mathbb{R}}|f_i(x)|\,dx = \infty$, the inequality is trivial. Let's assume $f_i \in L^1(\mathbb{R})$ for all i, we then fix an $\epsilon > 0$, set $a_i(x) = |f_i(x)|/(\int_{\mathbb{R}}|f_i(x)|\,dx + \epsilon)$ and integrate $\frac{1}{n}\sum_{i=1}^n a_i(x) \geq (\prod_{i=1}^n a_i(x))^{1/n}$ both sides over $\mathbb{R}$, we then get:

$$(1\geq)\ \frac{1}{n}\sum_{i=1}^n \frac{\int_{\mathbb{R}}|f_i|\,dx}{\int_{\mathbb{R}}|f_i|\,dx + \epsilon} \geq \frac{\int_{\mathbb{R}}(\prod_{i=1}^n |f_i|)^{1/n}\,dx}{\left(\prod_{i=1}^n(\int_{\mathbb{R}}|f_i|\,dx + \epsilon)\right)^{1/n}}$$

and (2.2) can be proved by setting $\epsilon \to 0$. In the proof the term ϵ is to avoid the case that $\int_{\mathbb{R}} |f_i(x)|\,dx = 0$. Its useful discrete analogue (whose proof is exactly the same) is:

$$\prod_{i=1}^{n}\left(\sum_{j=1}^{k} a_{ij}\right) \geq \left(\sum_{j=1}^{k}(a_{1j}a_{2j}\cdots a_{nj})^{1/n}\right)^{n},$$

where $n, k \geq 1$.

(a) Let $p \in [1,\infty)$ be given. If $\int_{\mathbb{R}} |f(x)|\,dx = \infty$, the inequality is trivial. Assume now $f \in L^1(\mathbb{R})$, by applying (2.2) once:

$$\begin{aligned}\left(\int_{\mathbb{R}} |g(y)||f(x-y)|\,dy\right)^p &\leq \int_{\mathbb{R}} |g^p(y)||f(x-y)|\,dy \left(\int_{\mathbb{R}} |f(x-y)|\,dy\right)^{p-1} \\ &= \|f\|_1^{p-1}\int_{\mathbb{R}} |g^p(y)||f(x-y)|\,dy,\end{aligned}$$

and since $\|f*g\|_p^p \leq \int_{\mathbb{R}}\left(\int_{\mathbb{R}} |g(y)||f(x-y)|\,dy\right)^p dx$,

$$\|f*g\|_p^p \leq \|f\|_1^{p-1}\int_{\mathbb{R}}\int_{\mathbb{R}} |g^p(y)||f(x-y)|\,dydx = \|f\|_1^p\|g\|_p^p,$$

here the change of order is valid by Fubini's Theorem (which can be done if one of $\iint |f(x,y)|\,dxdy$ and $\iint |f(x,y)|\,dydx$ is finite), so we are done.

When $p = \infty$, one has

$$|f*g(x)| \leq \int_{\mathbb{R}} |f(x-y)g(y)|\,dy \leq \int_{\mathbb{R}} |f(x-y)|\,dy\|g\|_\infty = \|f\|_1\|g\|_\infty,$$

as it is true for all $x \in \mathbb{R}$,

$$\|f*g\|_\infty =: \sup_{x\in\mathbb{R}} |f*g(x)| \leq \|f\|_1\|g\|_\infty.$$

(b) As $|x| \to \infty$, $|f(x)| \to 0$, there must be $x_0 \in \mathbb{R}$ such that $f(x_0) = \sup_{x\in\mathbb{R}}|f(x)|$, this can be shown as follows: If $f \equiv 0$, done. Assume there is $x_1 \in \mathbb{R}$, $|f(x_1)| > 0$, then there is $\delta > 0$ so that $|x| > \delta \implies |f(x)| < |f(x_1)|$ (clearly $|x_1| \leq \delta$), and $\sup f([-\delta,\delta]) = \max f([-\delta,\delta]) = |f(x_0)| = \|f\|_\infty$, for some $x_0 \in [-\delta,\delta]$.

Now the problem is readily solved. Let $|f(x_0)| = \sup|f(\mathbb{R})|$. Set $n = 2$ in (2.2), the celebrated Cauchy-Schwarz inequality, with $f_1 = f^2\chi_{[x_0,\infty)}$ and $f_2 = f'^2\chi_{[x_0,\infty)}$, then

$$\begin{aligned}\int_{x_0}^{\infty} |f(x)|^2\,dx \int_{x_0}^{\infty} |f'(x)|^2\,dx &= \int_{\mathbb{R}} |f_1|\,dx\int_{\mathbb{R}} |f_2|\,dx \\ &\geq \left(\int_{x_0}^{\infty} |f(x)||f'(x)|\,dx\right)^2 \\ &\geq \left(\int_{x_0}^{\infty} f(x)f'(x)\,dx\right)^2 \\ &= \frac{\|f\|_\infty^4}{4},\end{aligned}$$

and of course $\int_{\mathbb{R}} \cdot \geq \int_{x_0}^{\infty} \cdot$, we are done.

87. To be added.

88. Integrate $f'(t) = 6t + \sqrt{2+t^2}\sin^2 t$ from $t = 0$ to $t = x$, we have $f(x) = 5 + 3x^2 + \int_0^x \sqrt{2+t^2}\sin^2 t\,\mathrm{d}t$. Since the last integral is an odd function in x, it follows that

$$\int_{-2}^{2} f(x)\,\mathrm{d}x = \int_{-2}^{2}(5+3x^2)\,\mathrm{d}x + \int_{-2}^{2}\left(\int_0^x \sqrt{2+t^2}\sin^2 t\,\mathrm{d}t\right)\mathrm{d}x = 20 + 2^4 + 0 = 36.$$

89. The series converges since

$$\frac{1}{n(n+1)\cdots(n+k)} < \frac{1}{n^2(k-1)^2}.$$

We let $\frac{1}{x(x+1)\cdots(x+k)} = \sum_{r=0}^{k}\frac{a_r}{x+r}$, then $1 = \sum_{r=0}^{k} a_r \prod_{\substack{0\le j\le k\\ j\ne r}}(x+j)$, for all $x \in \mathbb{R}$. By choosing suitable x we can deduce that $1 = a_r(-1)^r r!(k-r)! \iff a_r = \frac{(-1)^r\binom{k}{r}}{k!}$, which implies

$$\frac{1}{n(n+1)\cdots(n+k)} = \frac{1}{k!}\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^r}{n+r}. \tag{1}$$

Since desired answer is an integral, for this end by binomial expansion and integration,

$$\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^r}{n+r} = (-1)^n\int_0^{-1}(1+x)^k x^{n-1}\,dx. \tag{2}$$

Combining (1) and (2), we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(n+1)\cdots(n+k)} &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\left(\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^r}{n+r}\right)\\
&= \sum_{n=1}^{\infty}(-1)^n\sum_{k=1}^{\infty}\int_0^{-1}\frac{1}{k!}(1+x)^k x^{n-1}\,dx\\
&\overset{(*)}{=} \sum_{n=1}^{\infty}(-1)^n\int_0^{-1}\sum_{k=1}^{\infty}\frac{1}{k!}(1+x)^k x^{n-1}\,dx\\
&= \sum_{n=1}^{\infty}\int_0^{1}(e^x-1)(1-x)^{n-1}\,dx\\
&\overset{(**)}{=} \int_0^{1}\sum_{n=1}^{\infty}(e^x-1)(1-x)^{n-1}\,dx.
\end{aligned}$$

(*) is true because $\left|\frac{(1+x)^k x^{n-1}}{k!}\right| \le \frac{1}{k!}$. To see (**) is true, it is *natural* to consider the inequality $e^x \ge 1+x \implies e^{-x} \ge 1-x$. Before we can take reciprocal, we make sure that right hand side is positive, so we choose $x \in [0,1)$, then a very nice upper bound $e^x - 1 \le \dfrac{x}{1-x}$ is obtained, for $x \in [0,1)$, so

$$(e^x-1)(1-x)^{n-1} \le \frac{x}{1-x}(1-x)^{n-1} = x(1-x)^{n-2}, \forall x \in [0,1).$$

We note that this inequality is also true for $x = 1$, so the above inequality holds for all $x \in [0,1]$.

Let $f_n(x) = \sum_{k=1}^{n}(e^x - 1)(1-x)^{k-1}$, clearly $f_n(0) = 0$ and when $x \neq 0$, we have

$$\begin{aligned} |f_n(x)| &= \left| \sum_{k=2}^{n}(e^x - 1)(1-x)^{k-1} + (e^x - 1) \right| \\ &\leq \left| \sum_{k=2}^{\infty} x(1-x)^{k-2} \right| + |e - 1| \\ &= e. \end{aligned}$$

This shows that $\{f_n\}$ is uniformly bounded by e and $\lim_{n\to\infty} f_n$ exists for each x (pointwise convergence is justified), so from **bounded convergence theorem** (which will be mentioned latter on in measure theory),

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \int_0^1 \lim_{n\to\infty} f_n(x)\, dx,$$

that is equivalent to saying that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \int_0^1 (e^x - 1)(1-x)^{k-1}\, dx = \int_0^1 \frac{e^x - 1}{x}\, dx.$$

90. Define $g(x) = \left(\int_a^x f(t)\, dt\right)^2 - \int_a^x f(t)^3\, dt$, differentiate once, we have

$$\begin{aligned} g'(x) &= 2f(x) \int_a^x f(t)\, dt - f(x)^3 \\ &= f(x) \left(2\int_a^x f(t)\, dt - \int_a^x d(f(t)^2) \right) \\ &= 2f(x) \int_a^x f(t)(1 - f'(t))\, dt. \end{aligned}$$

Since $f'(x) \geq 0 \implies f(x) \geq f(a) = 0$, we have $g'(x) \geq 0$, and hence

$$g(x) \geq g(a) = 0.$$

Thus the inequality follows.

91. Splitting the integral into two parts, we have

$$\int_0^1 \frac{h}{h^2 + x^2} f(x)\, dx = \left(\int_0^\delta + \int_\delta^1 \right) \frac{h}{h^2 + x^2} f(x)\, dx.$$

For any $\epsilon > 0$, there exists a δ such that $|x - 0| < \delta \implies |f(x) - f(0)| < \frac{\epsilon}{\frac{\pi}{2}}$. We now fix this δ, it follows that

$$\left| \int_0^\delta \frac{h}{h^2 + x^2} f(x)\, dx - \int_0^\delta \frac{h}{h^2 + x^2} f(0)\, dx \right| \leq \int_0^\delta \frac{h}{h^2 + x^2} |f(x) - f(0)|\, dx \leq \frac{\epsilon}{\frac{\pi}{2}} \int_0^\delta \frac{h}{h^2 + x^2}\, dx = \frac{\epsilon}{\frac{\pi}{2}} \tan^{-1}\left(\frac{\delta}{h}\right) <$$

Also there is $\sigma > h > 0$,

$$\left| \int_0^\delta \frac{h}{h^2 + x^2} f(0)\, dx - \frac{\pi}{2} f(0) \right| = |f(0)| \left| \tan^{-1} \frac{\delta}{h} - \frac{\pi}{2} \right| < \epsilon.$$

Since integrable functions are bounded, so there is $\sigma' > h > 0$,

$$\left| \int_\delta^1 \frac{h}{h^2 + x^2} f(x)\, dx \right| \leq \left| \int_\delta^1 \frac{h}{\delta^2} \sup_{x\in[0,1]} |f(x)| \right| = \frac{h(1-\delta)}{\delta^2} \sup_{x\in[0,1]} |f(x)|\, dx < \epsilon.$$

So adding these 3 inequalities when $h < \min\{\sigma, \sigma'\}$, we have

$$\left|\int_0^1 \frac{h}{h^2+x^2} f(x)\,dx - \frac{\pi}{2} f(0)\right|$$
$$\leq \left|\int_0^\delta \frac{h}{h^2+x^2} f(x)\,dx - \int_0^\delta \frac{h}{h^2+x^2} f(0)\,dx\right| + \left|\int_0^\delta \frac{h}{h^2+x^2} f(0)\,dx - \frac{\pi}{2} f(0)\right| + \left|\int_\delta^1 \frac{h}{h^2+x^2} f(x)\,dx\right| < 3\epsilon.$$

92. We prove this by Sandwich Theorem, since $f(x)$ is continuous on $[a,b]$, it must attend its maximum value in this interval, we denote this value as M, then

$$\left(\int_a^b f(x)^n\,dx\right)^{\frac{1}{n}} \leq M(b-a)^{\frac{1}{n}}. \quad (*)$$

On the other hand, there exists at least one $c \in [a,b]$ such that $f(c) = M$. As $f(x)$ is continuous at c, for any given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x-c| < \delta \implies |f(x) - M| < \epsilon.$$

Now we fix this δ and find that

$$\left(\int_a^b f(x)^n\,dx\right)^{\frac{1}{n}} \geq \left(\int_{c-\delta}^{c+\delta} f(x)^n\,dx\right)^{\frac{1}{n}} > (M-\epsilon)(2\delta)^{\frac{1}{n}}. \quad (**)$$

Combining (*) and (**) and letting $n \to \infty$, we have

$$M - \epsilon < \lim_{n\to\infty}\left(\int_a^b f(x)^n\,dx\right)^{\frac{1}{n}} \leq M < M + \epsilon$$
$$\implies \left|\lim_{n\to\infty}\left(\int_a^b f(x)^n\,dx\right)^{\frac{1}{n}} - M\right| < \epsilon.$$

we are done.

93. For any $a \geq -1$ and $n \in \mathbb{N}$, we have $(1+a)^n \geq 1+na$. We can prove this by induction on n or by the fact that $f(x) = (1+x)^n$ is convex when $n \geq 2$ such that $f(x) \geq f(0) + f'(0)(x-0) = 1+nx$. It follows that

$$\int_{-1}^1 (1-x^2)^n\,dx \geq \int_{-1}^1 (1-nx^2)\,dx \geq \int_{-\frac{1}{\sqrt{n}}}^{\frac{1}{\sqrt{n}}} (1-nx^2)\,dx = \frac{4}{3\sqrt{n}}.$$

94.

$$[x] \leq x < [x]+1 \implies a[x] < ax < a[x]+a,$$
$$[ax] \leq ax < [ax]+1 \implies -[ax]-1 < -ax < -[ax],$$

sum up these two, we have

$$a[x] - [ax] - 1 < 0 < -[ax] + a[x] + a$$
$$\implies -1 < [ax] - a[x] < a,$$

hence for any $x > 0$, we have $|[ax] - a[x]| < \max\{1, a\} = b$. By the substitution $x = \frac{1}{u}$,

$$\int_0^1 \left(\left[\frac{a}{x}\right] - a\left[\frac{1}{x}\right]\right) dx = \int_1^\infty ([au] - a[u])\frac{du}{u^2},$$

since

$$\int_1^\infty |[ax] - a[x]| \frac{1}{x^2}\,dx < \int_1^\infty b \cdot \frac{dx}{x^2},$$

the convergence of $\int_1^\infty b \cdot \frac{dx}{x^2}$ implies the convergence of $\int_1^\infty |[ax] - a[x]|\,dx$, thus $\int_1^\infty ([ax] - a[x])\,dx$ converges.

95. (a) The inequalities $e^{-x^2} \geq 1 - x^2$ and $e^{x^2} \geq 1 + x^2 \implies e^{-x^2} \leq \dfrac{1}{1+x^2}$ are obvious. Since $\lim_{t\to\infty} I(t)$ exists, the sequence $I(n)$ also converges to the same limit (this can be proved by simple ϵ-N reasoning). Now

$$I_n = \sqrt{n}\int_0^1 e^{-(\sqrt{n}x)^2}\,dx \geq \sqrt{n}\int_0^1 (1-x^2)^n\,dx \xlongequal{x=\sin\theta} \sqrt{n}\int_0^{\pi/2} \cos^{2n+1}\theta\,d\theta$$

$$\xlongequal{\text{integration by parts}} \sqrt{\frac{n}{2n+1}\cdot\frac{1}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2}. \tag{*}$$

$$I_n = \sqrt{n}\int_0^1 e^{-(\sqrt{n}x)^2}\,dx \leq \sqrt{n}\int_0^1 \frac{1}{(1+x^2)^n}\,dx$$

$$\underset{\int_0^{\pi/4}<\int_0^{\pi/2}}{\overset{x=\tan\theta}{<}} \sqrt{n}\int_0^{\pi/2}\cos^{2n-2}\theta\,d\theta = \sqrt{\frac{n}{2n-1}(2n-1)\left(\frac{(2n-3)!!}{(2n-2)!!}\right)^2\cdot\frac{\pi}{2}}. \tag{**}$$

When $n \to \infty$, combining (*) and (**) and by Wallis's formula, we have

$$\frac{\sqrt{\pi}}{2} \leq \lim_{\lambda\to\infty}\int_0^{\lambda} e^{-x^2}\,dx \leq \frac{\sqrt{\pi}}{2}.$$

96. In solving first order ODE, we are introduced a tool, integrating factor, we now bring this tool into play, having

$$\int_0^1 |f(x)-f'(x)|\,dx = \int_0^1 e^x\left|\frac{d}{dx}\left(e^{-x}f(x)\right)\right|\,dx \geq \int_0^1\left|\frac{d}{dx}\left(e^{-x}f(x)\right)\right|\,dx \geq \left|\int_0^1 d\left(e^{-x}f(x)\right)\right| = e^{-1}.$$

97. Since both x^2 and $f(x)^2$ are positive, we have $f'(x) > 0$, this shows that $f(x)$ is strictly increasing and hence we are left to show that $f(x)$ is bounded. In other words, showing $\lim_{x\to\infty} f(x) \leq 1 + \dfrac{\pi}{4}$ implies the first stuff we are asked to prove.

The fact that $f'(x) > 0$ implies the fact that $f(x) > f(1) = 1$, it follows that $f'(x) < \dfrac{1}{x^2+1}$, integrating both sides, we have

$$f(x) - f(1) < \int_1^x \frac{1}{y^2+1}\,dy < \int_1^\infty \frac{1}{y^2+1}\,dy = \frac{\pi}{2} - \frac{\pi}{4},$$

transposing terms, we get

$$f(x) < 1 + \frac{\pi}{4}, \forall x \geq a \implies \lim_{x\to\infty} f(x) \leq 1 + \frac{\pi}{4}.$$

98. In exactly the same way we are able to deduce that for any (integrable) decreasing function $f(x), g(x) \in \mathbb{R}$,

$$\int_a^b f(x)g(x)\,dx \geq \frac{1}{b-a}\int_a^b f(x)\,dx\int_a^b g(x)\,dx,$$

applying this once, we have

$$\int_a^b (-x)(-f(x))\,dx \geq \frac{1}{b-a}\cdot\frac{b^2-a^2}{2}\int_a^b f(x)\,dx = \left(\frac{a+b}{2}\right)\int_a^b f(x)\,dx.$$

99. Note that $\sum f(x_i^*)g(x_i^{**})\Delta x_i = \sum \int_{x_{i-1}}^{x_i} f(x_i^*)g(x_i^{**})\,dx$, we have

$$\begin{aligned}
&\left|\sum f(x_i^*)g(x_i^{**})\Delta x_i - \int_a^b f(x)g(x)\,dx\right| \\
\le& \sum \int_{x_{i-1}}^{x_i} |f(x_i^*)g(x_i^{**}) - f(x)g(x)|\,dx \\
\le& \sum \int_{x_{i-1}}^{x_i} \left(|f(x_i^*)g(x_i^{**}) - f(x_i^*)g(x)| + |f(x_i^*)g(x) - f(x)g(x)|\right)\,dx \\
=& \sum \int_{x_{i-1}}^{x_i} \left(|f(x_i^*)||g(x_i^{**} - g(x)| + |g(x)||f(x_i^*) - f(x)|\right)\,dx \\
\le& \sum \int_{x_{i-1}}^{x_i} \big(\omega_{[x_{i-1},x_i]}(g)|f(x_i^*)| + \omega_{[x_{i-1},x_i]}(f)|g(x)|\big)\,dx.
\end{aligned}$$

The fact that $f(x)$ and $g(x)$ are integrable implies that there exist M and N such that

$$|f(x)| < M \quad \text{and} \quad |g(x)| < N,$$

and for any $\epsilon > 0$, there exists a δ such that

$$\|P\| < \delta \implies \sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i, \sum \omega_{[x_{i-1},x_i]}(g)\Delta x_i < \frac{\epsilon}{M+N}.$$

Hence, we conclude that

$$\left|\sum f(x_i^*)g(x_i^{**})\Delta x_i - \int_a^b f(x)g(x)\,dx\right| \le \frac{\epsilon}{M+N}(M+N) = \epsilon.$$

100. For any $x \in [0,1]$, we see that

$$\big(M - f(x)\big)\left(\frac{1}{m} - \frac{1}{f(x)}\right) \ge 0 \iff \frac{M}{m} + 1 \ge \frac{f(x)}{m} + \frac{M}{f(x)},$$

integrate both sides from 0 to 1, and use AM-GM inequality on right hand side once, we are done.

101. Define $f(t) = \int_0^t |x(u)|\,du$, since $x(t)$ is continuous $\implies$ $|x(t)|$ is continuous, thus the integral defined in f makes sense. By the continuity of $|x(u)|$, we have $f'(t) = x(t)$, thus

$$f'(t) \le M + kf(t) \iff \frac{d}{dt}\big(e^{-kt}f(t)\big) \le Me^{-kt} \implies kf(t) \le M(e^{kt} - 1),$$

it follows that

$$|x(t)| \le M + kf(t) \le Me^{kt}.$$

102. Since $\int_{-1}^1 f(x)g(x)\,dx = \left(\int_{-1}^0 + \int_0^1\right) f(x)g(x)\,dx$, by the change of variable $y = -x$ to the former integral, we have

$$\int_0^1 \big(f(x) + f(-x)\big)g(x)\,dx = 0,$$

for any g such that the integral makes sense.

For the sake of contradiction, suppose there is an $a \in [0,1]$ such that $f(a) + f(-a) \ne 0$, it is no loss of generality to suppose that $f(a) + f(-a) > 0$, then due to continuity of $f(x) + f(-x)$, there exists a $\delta > 0$ such that

$$x \in [a - \delta, a + \delta] \implies f(x) + f(-x) > 0.$$

Now $g(x)$ is arbitrary if it doesn't affect integrability (that means it can't be too crazy), we are free to take $g(x)$ satisfying

$$\big(f(x)+f(-x)\big)g(x)\begin{cases}>0 & x\in[a-\delta,a+\delta]\\ \ge 0 & x\in[0,1]-[a-\delta,a+\delta]\end{cases}$$

then

$$\int_0^1 \big(f(x)+f(-x)\big)g(x)\,dx \ge \int_{a-\delta}^{a+\delta}\big(f(x)+f(-x)\big)g(x)\,dx>0,$$

a contradiction, thus $f(-x)=f(x)$, for all $x\in[0,1]$.

Remark. *Any funny setting of $g(x)$ is fine once you can derive any contradiction. For example, we can take $g(x)>0$ when $x\in[a-\delta,a+\delta]$ and $g(x)=0$ when $x\in[0,1]-[a-\delta,a+\delta]$. Then the integral will also be bigger than zero, same contradiction arises.*

103. Simple computation gives us $\int_{-\pi}^0|\sin x|\,dx=2$ and we observe that

$$\int_0^\pi f(x)|\sin nx|\,dx=\int_{-\pi}^0\frac{1}{n}\sum_{k=1}^n f\left(\frac{x}{n}+\frac{k\pi}{n}\right)|\sin x|\,dx,$$

this can be shown by the change of variable $x=y/n$ and breaking the integral $\int_0^{n\pi}=\sum_{k=1}^n\int_{(k-1)\pi}^{k\pi}$. Finally, we shift the integrand and the function to the left by $k\pi$ unit.

Now we can "feel" that $\frac{1}{n}\sum_{k=1}^n f\underbrace{\left(\frac{x}{n}+\frac{k\pi}{n}\right)}_{x_n}\approx\frac{1}{n}\sum_{k=1}^n f\underbrace{\left(\frac{k\pi}{n}\right)}_{y_n}\approx\int_0^1 f(x\pi)\,dx$ when n is large, we expect

$$\lim_{n\to\infty}\int_{-\pi}^0\frac{1}{n}\sum_{k=1}^n f\left(\frac{x}{n}+\frac{k\pi}{n}\right)|\sin x|\,dx\sim\int_{-\pi}^0\left(\int_0^1 f(x\pi)\,dx\right)|\sin x|\,dx=\frac{2}{\pi}\int_0^\pi f(x)\,dx.$$

We now prove that "$\sim$" is actually a "$=$".

Since f is continuous on a closed bounded interval (a compact set), it is uniformly continuous. For any $\epsilon>0$, there is a δ such that $|x_k-y_k|<\delta\implies|f(x_k)-f(y_k)|<\frac{\epsilon}{2\pi}$. We see that $|x_k-y_k|=\frac{|x|}{n}$. So for large enough N_1,

$$n>N_1\implies|x_k-y_k|<\delta\implies|f(x_k)-f(y_k)|<\frac{\epsilon}{4}.$$

As a result, when $n>N_1$,

$$\left|\int_{-\pi}^0\frac{1}{n}\sum_{k=1}^n f(x_k)|\sin x|\,dx-\int_{-\pi}^0\frac{1}{n}\sum_{k=1}^n f(y_k)|\sin x|\,dx\right|<\frac{\epsilon}{4}\int_{-\pi}^0\frac{1}{n}\sum_{k=1}^n|\sin x|\,dx=\frac{\epsilon}{2}.\quad(1)$$

On the other hand, as continuity implies integrability on a closed bounded interval, so for any $\epsilon>0$, there is a N_2 such that $n>N_2\implies\left|\frac{1}{n}\sum_{k=1}^n f(y_k)-\int_0^1 f(x)\,dx\right|<\frac{\epsilon}{4}$, so when $n>N_2$,

$$\left|\int_{-\pi}^{0}\frac{1}{n}\sum_{k=1}^{n}f(y_k)|\sin x|\,dx-\int_{-\pi}^{0}\left(\int_{0}^{1}f(x\pi)\,dx\right)|\sin x|\,dx\right|<\frac{\epsilon}{4}\int_{-\pi}^{0}|\sin x|\,dx=\frac{\epsilon}{2}. \quad (2)$$

Adding inequalities (1) and (2) and applying triangle inequality once, we have

$$n>\max\{N_1,N_2\}\implies\left|\underbrace{\int_{-\pi}^{0}\frac{1}{n}\sum_{k=1}^{n}f(x_k)|\sin x|\,dx}_{=\int_0^\pi f(x)|\sin nx|\,dx}-\underbrace{\int_{-\pi}^{0}\left(\int_{0}^{1}f(x\pi)\,dx\right)|\sin x|\,dx}_{=\frac{2}{\pi}\int_0^\pi f(x)\,dx}\right|<\epsilon.$$

104. Let $c\in[0,1)$, $|f(x)|\le M$ for all $x\in[0,c]$, then

$$\left|n\int_0^c f(x)x^{2n}\,dx\right|\le nM\int_0^c x^{2n}\,dx=\frac{n}{2n+1}Mc^{2n+1},$$

hence

$$\lim_{n\to\infty}n\int_0^c f(x)x^{2n}\,dx=0,\forall c\in[0,1). \quad (*)$$

Finally, for any $\epsilon>0$, there a δ such that $1-\delta<x\le 1\implies|f(x)-f(1)|<2\epsilon$. Now

$$\begin{aligned}&\left|n\int_{1-\delta}^{1}f(x)x^{2n}\,dx-\frac{f(1)}{2}\right|\\ \le&\left|n\int_{1-\delta}^{1}f(x)x^{2n}\,dx-n\int_{1-\delta}^{1}f(1)x^{2n}\,dx\right|+\left|n\int_{1-\delta}^{1}f(1)x^{2n}\,dx-\frac{f(1)}{2}\right|\\ <&\,2\epsilon\frac{n}{2n+1}\left(1-(1-\delta)^{2n+1}\right)+\left|\frac{n}{2n+1}\left(1-(1-\delta)^{2n+1}\right)-\frac{1}{2}\right|f(1)\end{aligned}$$

now we take $n\to\infty$ on both sides, having

$$\left|\lim_{n\to\infty}n\int_{1-\delta}^{1}f(x)x^{2n}\,dx-\frac{f(1)}{2}\right|\le\epsilon. \quad (**)$$

We see that by (*) and (**),

$$\left|\lim_{n\to\infty}\left(\int_0^{1-\delta}+\int_{1-\delta}^{1}\right)f(x)x^{2n}\,dx-\frac{f(1)}{2}\right|\le 0+\epsilon=\epsilon,$$

but ϵ is arbitrarily small, we must have

$$\lim_{n\to\infty}n\int_0^1 f(x)x^{2n}\,dx=\frac{f(1)}{2}.$$

105. Denote $I_n=n^2\int_0^1(\sqrt[n]{1+x^n}-1)\,dx$, let $y=x^n$, then

$$I_n=n\int_0^1\frac{\sqrt[n]{1+y}-1}{y}y^{\frac{1}{n}}\,dy.$$

Now by the fact that for each fixed y there is a $c_{y,\frac{1}{n}}$ between 0 and $\frac{1}{n}$ such that (Taylor series of $(1+y)^x$ with remainder term, 3 terms in total)

$$0 < n(\sqrt[n]{1+y}-1) - \ln(1+y) = (1+y)^{c_{y,\frac{1}{n}}}\big(\ln(1+y)\big)^2\frac{1}{2n},$$

we have

$$\begin{aligned}\left|I_n - \int_0^1 \frac{\ln(1+y)}{y} y^{\frac{1}{n}}\,dy\right| &\le \frac{1}{2n}\int_0^1 \frac{(1+y)^{\frac{1}{n}}\big(\ln(1+y)\big)^2}{y} y^{\frac{1}{n}}\,dy \\ &\le \frac{1}{2n}\int_0^1 (1+y)^{\frac{1}{n}} y^{\frac{1}{n}+1}\,dy \\ &\le \frac{1}{2n}\int_0^1 \left(\frac{1}{n}y+1\right) y^{\frac{1}{n}+1}\,dy \\ &\le \frac{1}{2n}\left(\frac{1}{n}+1\right)\int_0^1 y^{\frac{1}{n}+1}\,dy \\ &= \frac{n+1}{2n(2n+1)},\end{aligned}$$

here we have used the fact that $\ln(1+y) \le y$ and $(1+y)^{\frac{1}{n}} \le \frac{1}{n}y+1$ (then do direct integration).

Finally we can prove that

$$\left|\int_0^1 \frac{\ln(1+y)}{y} y^{\frac{1}{n}}\,dy - \int_0^1 \frac{\ln(1+y)}{y}\,dy\right| \le \int_0^1 (1-y^{\frac{1}{n}})\,dy = \frac{1}{n+1},$$

then by adding the two inequalies, we get

$$\left|I_n - \int_0^1 \frac{\ln(1+y)}{y}\,dy\right| \le \frac{n+1}{2n(2n+1)} + \frac{1}{n+1},$$

this inequality shows that $\lim_{n\to\infty} I_n$ exists and the limit is $\int_0^1 \frac{\ln(1+y)}{y}\,dy$. The evaluation of $\int_0^1 \frac{\ln(1+y)}{y}\,dy$ makes use of uniform convergence. Here is a sketch, first express $\ln x$ in power series, do termwise integration, the resulting series has a well-known limit.

106. We first deal with the integral, by geometric fact,

$$\int_k^{k+1} x\ln\big((x-k)(k+1-x)\big)\,dx = \int_0^1 (x-\tfrac{1}{2}+k+\tfrac{1}{2})\ln\big(x(1-x)\big)\,dx.$$

Observing that $\displaystyle\int_0^1 (x-\tfrac{1}{2})\ln\big(x(1-x)\big)\,dx = \int_{-1/2}^{1/2} x\log(\tfrac{1}{4}-x^2)\,dx = 0$, we have

$$\int_k^{k+1} x\ln\big((x-k)(k+1-x)\big)\,dx = (k+\tfrac{1}{2})\int_0^1 \ln\big(x(1-x)\big)\,dx.$$

We can evaluate the last integral by first finding its anti-derivative and taking limit, but we have an aha! way to compute this instead of mundane method. Note that any function is symmetric about $y=x$ with its inverse function. In our case, we need the inverse function of $f(x)=\ln x$, namely, $f^{-1}(x)=e^x$, hence

$$\int_0^1 \ln\big(x(1-x)\big)\,dx = 2\int_0^1 \ln x\,dx = -2\int_{-\infty}^0 e^x\,dx = -2.$$

Plugging in all results in the original sum, and by the particular partition of Riemann sum, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left\{\frac{1}{n^4}\left(\sum_{k=1}^{n}k^2\int_k^{k+1}x\ln\big((x-k)(k+1-x)\big)\,\mathrm{d}x\right)\right\}\\
&=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(\frac{k}{n}\right)^2\frac{1}{n}(k+\tfrac{1}{2})(-2)\\
&=-2\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(\frac{k}{n}\right)^3\\
&=-\frac{1}{2}.
\end{aligned}$$

107. To be added.

2.3 Evaluation of Limit

108. (a) We mainly use the fact that $f(x) = T_n(x) + o((x - x_0)^n)$. Consider

$$y = \prod_{k=2}^{n}(\cos kx)^{\frac{1}{k}},$$

then when x close to 0, we have

$$\cos x = 1 - \frac{x^2}{2!} + o(x^2) \quad \text{and} \quad \log(1 + x) = x + o(x)$$

and hence

$$\begin{aligned}
\log y &= \sum_{k=2}^{n}\frac{1}{k}\log\cos kx = \sum_{k=2}^{n}\frac{1}{k}\left(-\frac{k^2x^2}{2}+o(x^2)\right)\\
&=\sum_{k=2}^{n}\frac{-kx^2}{2}+\sum_{k=2}^{n}\frac{1}{k}o(x^2)\\
&=\sum_{k=2}^{n}\frac{-kx^2}{2}+o(x^2),
\end{aligned}$$

so

$$\begin{aligned}
\frac{y-1}{x^2} &= \frac{\exp(\sum_{k=2}^{n}\frac{-kx^2}{2}+o(x^2))-1}{x^2} = \frac{(1+\sum_{k=2}^{n}\frac{-kx^2}{2}+o(x^2))-1}{x^2}\\
&=-\frac{1}{2}\sum_{k=2}^{n}k+\frac{o(x^2)}{x^2}\\
\lim_{x\to0}\frac{y-1}{x^2} &= -\frac{1}{2}\sum_{k=2}^{n}k+\lim_{x\to0}\frac{o(x^2)}{x^2} = \frac{(n+2)(1-n)}{4}.
\end{aligned}$$

(b) Since $\prod_{k=2}^{n}\sqrt[k]{1-kx}$ is continuous at 0 and hence the limit is of $\frac{0}{0}$ type, L'hôspital rule can be applied.

$$\lim_{x\to 0}\frac{\prod_{k=2}^{n}\sqrt[k]{1-kx}-1}{\sin x}=\lim_{x\to 0}\frac{\sum_{r=2}^{n}-\frac{1}{(1-rx)^{1-\frac{1}{r}}}\prod_{k\neq r}(1-kx)^{\frac{1}{k}}}{\cos x}$$
$$=-\sum_{r=2}^{n}(1)$$
$$=1-n.$$

(c) To be added.

(d) We use Taylor expansion with remainder term (just showing the order of infinitesimal).

$$I_n=n\left(\frac{1^\alpha+2^\alpha+...+n^\alpha}{n^{\alpha+1}}-\frac{1}{\alpha+1}\right)=\frac{(\alpha+1)(1^\alpha+\cdots+n^\alpha)-n^{\alpha+1}}{n^\alpha(\alpha+1)}=\frac{h_n}{k_n}.$$

Let

$$J_n=\frac{\Delta h_n}{\Delta k_n}=\frac{(\alpha+1)n^\alpha-n^{\alpha+1}+(n-1)^{\alpha+1}}{(\alpha+1)\big(n^\alpha-(n-1)^\alpha\big)}=\frac{(\alpha+1)-n+(1-\frac{1}{n})^\alpha(n-1)}{(\alpha+1)\big(1-(1-\frac{1}{n})^\alpha\big)}.$$

Recall that $(1+x)^\alpha=1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+o(x^2)$, note that we also have

$$\lim_{n\to\infty}\frac{(n-1)o(\frac{1}{n^2})}{\frac{1}{n}}=0\implies (n-1)o\left(\frac{1}{n^2}\right)=o\left(\frac{1}{n}\right),$$

hence

$$\lim_{n\to\infty}J_n=\lim_{n\to\infty}\frac{\alpha+\frac{\alpha(\alpha-1)}{2}\frac{n-1}{n}+\frac{o(\frac{1}{n})}{\frac{1}{n}}}{(\alpha+1)\left(\alpha+\frac{o(\frac{1}{n})}{\frac{1}{n}}\right)}=\frac{\alpha+\frac{\alpha(\alpha-1)}{2}}{(\alpha+1)(\alpha)}=\frac{1}{2}\xlongequal{\text{Stolz theorem}}\lim_{n\to\infty}I_n.$$

109. Define $p_n=(1+a_1)(1+a_2)\cdots(1+a_n)$, then

$$\frac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)}=\frac{(a_n+1)-1}{p_n}=\frac{\frac{p_n}{p_{n-1}}-1}{p_n}=\frac{1}{p_{n-1}}-\frac{1}{p_n}.$$

We see that

$$\sum_{n=2}^{m}\frac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)}=\frac{1}{p_1}-\frac{1}{p_m}.$$

Since $p_n=(1+a_n)p_{n-1}>p_{n-1}\implies\{p_n\}$ is strictly increasing, we have only two possibilities

(a) $\lim_{n\to\infty}p_n=+\infty$ (e.g. $\prod_{n=1}^{\infty}(1+\frac{1}{n})$);

(b) $\lim_{n\to\infty}p_n<+\infty$ (e.g. $\prod_{n=1}^{\infty}(1+\frac{1}{n^2})$).

In both cases, we can conclude that $\sum_{n=2}^{\infty}\frac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)}$ converges, thus

$$\lim_{n\to\infty}\frac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)}=0.$$

110. We see that

$$\begin{aligned}
&\left|\frac{a_1b_n+\cdots+a_nb_1}{n}-ab\right| \\
&<\left|\frac{\sum_{k=1}^N a_kb_{n-k+1}-ab}{n}\right|+\frac{1}{n}\sum_{k=N+1}^{n}|a_kb_{n-k+1}-ab_{n-k+1}+ab_{n-k+1}-ab| \\
&<\left|\frac{\sum_{k=1}^N a_kb_{n-k+1}-ab}{n}\right|+\frac{\sum_{k=N+1}^n |b_{n-k+1}||a_k-a|}{n}+\frac{\sum_{k=N+1}^{n-N} a|b_{n-k+1}-b|}{n}+\frac{\sum_{k=n-N+1}^{n} a|b_{n-k+1}-b|}{n}
\end{aligned}$$

Now N is suitably chosen such that

$$n>N \implies \begin{cases} \dfrac{1}{n}\left|\sum\limits_{k=1}^{N}(a_kb_{n-k+1}-ab)\right|<\epsilon \\ |a_n-a|<\epsilon \\ |b_n-b|<\epsilon \\ \dfrac{1}{n}\sum\limits_{k=n-N+1}^{n} a|b_{n-k+1}-b|<\epsilon \end{cases}.$$

Morever, $\{b_n\}$ converges implies that $\{b_n\}$ is a bounded sequence, suppose that $|b_n|<M$, for any $b\in\mathbb{N}$, then it follows that

$$\left|\frac{a_1b_n+\cdots+a_nb_1}{n}-ab\right|<\epsilon+\frac{n-N}{n}M\epsilon+\frac{n-2N}{n}a\epsilon+\epsilon<(M+a+2)\epsilon.$$

111. To be added.

112. Notice that $a_n=\sqrt{1+a_2+\cdots+a_{n-1}}>1$, for all $n\ge 2$, so $a_{n+1}>\sqrt{n}$, thus $\lim_{n\to\infty}a_n=\infty$. Note that the recurrence relation can be written as $a_{n+1}^2=a_n^2+a_n$, so $\frac{a_{n+1}}{a_n}\to 1$, and by Stolz theorem,

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{(n+1)-n}=\lim_{n\to\infty}\frac{1}{1+\frac{a_{n+1}}{a_n}}=\frac{1}{2}=\lim_{n\to\infty}\frac{a_n}{a_n}.$$

113. Let $N\in\mathbb{N}$ such that

$$n>N \implies \begin{cases} |t_{nk}|<\epsilon \\ |a_n-a|<\epsilon \end{cases}.$$

Since $\{a_n\}$ is a convergent sequence, there exists a positive M such that $|a_n-a|<M,\forall n\in\mathbb{N}$. Then

$$\begin{aligned}
|x_n-a| &= \left|\sum_{k=1}^{n}t_{nk}a_k-\sum_{k=1}^{n}t_{nk}a\right|=\left|\left(\sum_{k=1}^{N}+\sum_{k=N+1}^{n}\right)t_{nk}(a_k-a)\right| \\
&<\sum_{k=1}^{N}|t_{nk}||a_k-a|+\sum_{k=N+1}^{n}|t_{nk}||a_k-a| \\
&<MN\epsilon+\epsilon=(MN+1)\epsilon.
\end{aligned}$$

114. Observe that

$$\left|\frac{\sum_{k=1}^n ka_k}{n^2}-\frac{a}{2}\right|<\left|\frac{\sum_{k=1}^N ka_k}{n^2}\right|+\left|\frac{\sum_{k=N+1}^n k(a_k-a)}{n^2}\right|+\left|\frac{\sum_{k=N+1}^n ka}{n^2}-\frac{a}{2}\right|.$$

Since each term on right hand side is arbitrarily small. we are done.

On the other hand, we can show the result by stolz theorem, since $\{n^2\}$ is strictly increasing and unbounded, we have

$$\lim_{n\to\infty}\frac{(a_1+2a_2+\cdots+na_n)-(a_1+2a_2+\cdots+(n-1)a_{n-1})}{n^2-(n-1)^2}=\lim_{n\to\infty}\frac{1}{2}\frac{n}{n-\frac{1}{2}}a_n=\frac{a}{2},$$

that shows desired result.

115. To be added.

116. By equivalence of norm, there exist two non-zero constants d_1, d_2 such that $d_1\|x\|_\infty \le \|x\| \le d_2\|x\|_\infty$, in other words, there exists two constants c_1, c_2 such that $c_1\frac{f(x,y)}{\|\vec{x}\|_\infty} \le \frac{f(x,y)}{\|\vec{x}\|} \le c_2\frac{f(x,y)}{\|\vec{x}\|_\infty}$, hence

$$\lim_{(x,y)\to\vec{0}}\frac{f(x,y)}{\|\vec{x}\|}=0 \iff \lim_{(x,y)\to\vec{0}}\frac{f(x,y)}{\|\vec{x}\|_\infty}=0.$$

The original limit exists and equals to 0 if and only if

$$\lim_{(x,y)\to\vec{0}}\frac{|x|^p|y|^q}{\max\{|x|,|y|^{q/k}\}^{\alpha k}\max\{|x|,|y|^{n/m}\}^{\beta m}}=0.$$

Now we try to study the limit case by case.

$$\lim_{(x,y)\to\vec{0}}\frac{|x|^p|y|^q}{\max\{|x|,|y|^{q/k}\}^{\alpha k}\max\{|x|,|y|^{n/m}\}^{\beta m}}=\begin{cases}|x|^p|y|^{q-l\alpha-n\beta} & \text{if } |y|^{n/m},|y|^{l/k}\ge|x|\\ |x|^{p-\alpha k-\beta m}|y|^q & \text{if } |x|\ge|y|^{n/m},|y|^{l/k}\\ |x|^{p-\beta m}|y|^{q-l\alpha} & \text{if } |y|^{n/m}\le|x|\le|y|^{l/m}\\ |x|^{p-\alpha k}|y|^{q-n\beta} & \text{if } |y|^{l/k}\le|x|\le|y|^{n/m}\end{cases}$$

Combining all results, we have quite a number of possibilities.

i) $\frac{l}{k}p+q-l\alpha-n\beta>0$	iv) $p-\alpha k-\beta m+\frac{k}{l}q>0$	vii) $\frac{n}{m}(p-\alpha k)+q-n\beta>0$
ii) $\frac{n}{m}p+q-l\alpha-n\beta>0$	v) $p-\beta m+\frac{m}{n}(q-l\alpha)>0$	
iii) $p-\alpha k-\beta m+\frac{m}{n}q>0$	vi) $\frac{l}{k}(p-\beta m)+q-l\alpha>0$	viii) $p-\alpha k+\frac{k}{l}(q-n\beta)>0$

117. To be added.

2.4 Sequence and Series

118. (a) First note that $\sqrt{k(k+4)+20}=(k+2)\sqrt{1+\left(4/(k+2)\right)^2}$, it conveys us some messages. As k increases, we have $\sqrt{k(k+4)+20}=k+2+f(k)$, where $f(k)\to 0$ as $k\to\infty$. But only the integral part deserves our concern, so if $k+2<\sqrt{k(k+4)+20}<k+3$, then everything goes smooth and we have $[\sqrt{k(k+4)+20}]=k+2$. But when does this inequality hold? Just solve the inequality we have $k>5.5$, so the inequality is true of $k\ge 6$ and thus the value should be $5+5+6+7+8+8+9+10+\cdots+102=5256$.

(b) $\sum_{k=1}^{n}[\sqrt{k}]=\sum_{j=1}^{a}\sum_{k=(j-1)^2}^{j^2-1}[\sqrt{k}]+\sum_{k=a^2}^{n}[\sqrt{k}]=\sum_{j=1}^{a}(2j-1)(j-1)+(n-a^2+1)a$. Finally we make use of the formulas $\sum_{j=1}^{n} j^2 = \frac{n}{6}(n+1)(2n+1)$ and $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$, having $\sum_{k=1}^{n}[\sqrt{k}] = \frac{1}{6}a(-2a^2 - 3a + 5 + 6n)$.

119. Denote $f(x) = u^3 + v^3 + w^3 - 3uvw$, to show that $f(x) = 1$, we first show that $f'(x) = 0$, then show that $f(x) = f(0) = 1$.

We know that $f(0) = 1$, it suffices to show that $f'(x) = 0$. Differentiating f once, we have

$$f'(x) = 3\sum_{cyc} u^2u' - 3\sum_{cyc} u'vw = 3\sum_{cyc} u'(u^2 - vw).$$

We also observe that $u' = w$, $v' = u$, $w' = v$, it follows that

$$\sum_{cyc} u'(u^2 - vw) = \sum_{cyc} w(u^2 - vw) = \sum_{cyc}(u^2w - vw^2) = \sum_{cyc} w^2v - \sum_{cyc} vw^2 = 0.$$

From complex analysis, we know that $e^z = \sum\limits_{n=0}^{\infty} z^n/n!, \forall z \in \mathbb{C}$. Now for any real x, we have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \left(\sum_{n\equiv 0 \pmod 3} + \sum_{n\equiv 1 \pmod 3} + \sum_{n\equiv 2 \pmod 3} \right) \frac{x^n}{n!},$$

$$\mathrm{Re}\,(e^{\omega x}) = \sum_{n\equiv 0 \pmod 3} \frac{x^n}{n!} - \frac{1}{2}\left(\sum_{n\equiv 1 \pmod 3} + \sum_{n\equiv 2 \pmod 3} \right) \frac{x^n}{n!} = \frac{3}{2}\sum_{n\equiv 0 \pmod 3} \frac{x^n}{n!} + e^x,$$

here ω is the cube root of unity. Transposing terms, we have

$$u = \sum_{n\equiv 0 \pmod 3} \frac{x^n}{n!} = \frac{1}{3}e^x + \frac{2}{3}e^{-x/2}\cos\frac{\sqrt{3}x}{2}.$$

120. Recall the fact that $\cos x = \frac{e^{ix} + e^{-ix}}{2}$, we have

$$\begin{aligned}
\sum_{k=0}^{n-1}(-1)^k \cos^n\left(\frac{k\pi}{n}\right) &= \sum_{k=0}^{n-1}(-1)^k \left(\frac{e^{ik\pi/n} + e^{-ik\pi/n}}{2}\right)^n \\
&= \frac{1}{2^n}\sum_{k=0}^{n-1}(-1)^k \sum_{r=0}^{n} \binom{n}{r} \left(-e^{i\pi(-2r)/n}\right)^k \\
&= \frac{1}{2^n}\sum_{k=0}^{n-1}\left(\sum_{r=1}^{n-1}\binom{n}{r}(e^{-2ri\pi/n})^k + 2\right) \\
&= \frac{1}{2^n}\left(\sum_{r=1}^{n-1}\binom{n}{r}\frac{1 - e^{-2ri\pi}}{1 - e^{-2ri\pi/n}} + 2n\right) \\
&= \frac{n}{2^{n-1}}.
\end{aligned}$$

121. (a), (b) By taylor expansion together with the remainder, we have

$$|x_n - \log(1 + x_n)| = \left|\frac{1}{2(1 + c_n)^2}x_n^2\right|, \text{ for some } c_n \text{ between 0 and } x_n. \qquad (*)$$

Since $e^x \geq 1+x$, for any $x \in \mathbb{R}$, we have $x_n - \log(1+x_n) \geq 0$, for any x_n. Moreover, either one of convergences of $\sum x_n$ and $\sum x_n^2$ implies that $\lim_{n\to\infty} x_n = 0$. Hence for given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n > N \implies |x_n| < \epsilon \implies |c_n| < \epsilon.$$

We require $\epsilon < 1$, then from (*), we have for $n > N$,

$$\frac{1}{2(1+\epsilon)^2}x_n^2 < x_n - \log(1+x_n) < \frac{1}{2(1-\epsilon)^2}x_n^2.$$

This inequality is just enough to prove the statements by comparison test.

122. **Method 1.** Integration goes some way to attacking this problem. A series $\sum u_n(x)$ is integrable and its anti-derivative converges to $\sum \int_a^b u_n(x)\,dx$ if $\sum u_n(x)$ is uniformly convergent on $[a,b]$.

Now since for any x, excluding the points at which $f(x)$ is not defined, $\lim_{n\to\infty} \tan\frac{x}{2^n} = 0$, hence there exists an N such that

$$n > N \implies \left|\frac{1}{2^k}\tan\frac{x}{2^k}\right| < \frac{1}{2^{k+1}}.$$

By comparison test, $\sum 1/2^{k+1}$ converges implies $\sum \frac{1}{2^n}\tan\frac{x}{2^n}$ converges absolutely, and hence $\sum \frac{1}{2^n}\tan\frac{x}{2^n}$ converges uniformly.

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{2^k}\int_0^x \tan\frac{t}{2^k}\,dt = -\sum_{k=1}^{\infty}\ln\cos\frac{x}{2^k} &= -\ln\left(\prod_{k=1}^{\infty}\cos\frac{x}{2^k}\right) = -\ln\left(\lim_{n\to\infty}\frac{\sin x}{2^n\sin\frac{x}{2^n}}\right)\\ &= -\ln\left[\left(\frac{\sin x}{x}\right)\left(\lim_{n\to\infty}\frac{\sin(x/2^n)}{x/2^n}\right)^{-1}\right] = \ln\frac{x}{\sin x},\end{aligned}$$

Now since $\sum_{k=1}^{\infty}\frac{1}{2^k}\int_0^x \tan\frac{t}{2^k}\,dt$ converges and, as proved, $\sum\frac{1}{2^n}\tan\frac{x}{2^n}$ converges uniformly, we differentiate both sides, yielding

$$\sum_{k=1}^{\infty}\frac{1}{2^k}\tan\frac{x}{2^k} = \frac{\sin x - x\cos x}{x\sin x} = \frac{1}{x} - \cot x.$$

Method 2. Observe that

$$\tan x - \cot x = -2\cot 2x,$$

replace x by $\frac{x}{2^k}$ and divide both sides by 2^k, that gives

$$\frac{1}{2^k}\tan\frac{x}{2^k} = \frac{1}{2^k}\cot\frac{x}{2^k} - \frac{1}{2^{k-1}}\cot\frac{x}{2^{k-1}},$$

it follows that

$$\sum_{k=1}^{\infty}\frac{1}{2^k}\tan\frac{x}{2^k} = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{2^k}\cot\frac{x}{2^k} - \frac{1}{2^{k-1}}\cot\frac{x}{2^{k-1}}\right) = \lim_{n\to\infty}\left(\frac{1}{x}\cos\frac{x}{2^n}\frac{\frac{x}{2^n}}{\sin\frac{x}{2^n}} - \cot x\right) = \frac{1}{x} - \cot x.$$

123. We use summation by parts,

$$\frac{\sum_{k=1}^{n} ka_k}{n} = -\frac{\sum_{k=1}^{n-1}\sum_{i=1}^{k} a_i}{n} + \sum_{i=1}^{n} a_i.$$

We also know that $\lim_{n\to\infty} a_n = L \implies \lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = L$, thus

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} ka_k}{n} = -\left(\lim_{n\to\infty} \frac{\sum_{k=1}^{n-1}\sum_{i=1}^{k} a_i}{n-1}\right)\left(\lim_{n\to\infty} \frac{n-1}{n}\right) + \lim_{n\to\infty} \sum_{i=1}^{n} a_i = 0.$$

124. We also use summation by parts,

$$\frac{\sum_{k=1}^{n} a_k b_k}{b_n} = \frac{\sum_{k=1}^{n-1}\sum_{i=1}^{k} a_i(b_k - b_{k+1}) + b_n \sum_{i=1}^{n}}{b_n}.$$

We rewrite b_n as $\sum\limits_{k=1}^{n-1}(b_{k+1} - b_k) + b_1$, we have

$$\frac{\sum_{k=1}^{n} a_k b_k}{b_n} = \frac{\sum_{k=1}^{n-1}(\sum_{i=1}^{k} a_i - \sum_{i=1}^{n} a_i)(b_k - b_{k+1})}{b_n} + \frac{b_1 \sum_{i=1}^{n} a_i}{b_n},$$

hence we have

$$\left|\frac{\sum_{k=1}^{n} a_k b_k}{b_n}\right| < \frac{1}{|b_n|}\left|\left(\sum_{k=1}^{N} + \sum_{k=N+1}^{n-1}\right)\sum_{i=k}^{n} a_i(b_k - b_{k+1})\right| + \left|\frac{b_1 \sum_{i=1}^{n} a_i}{b_n}\right|.$$

For given $\epsilon > 0$, there exists an integer $N \in \mathbb{N}$ such that

$$m, n > N \implies \begin{cases} \dfrac{1}{|b_n|}\left|\displaystyle\sum_{k=1}^{N}\sum_{i=k}^{n} a_i(b_k - b_{k+1})\right| < \dfrac{\epsilon}{3} \\ \left|\displaystyle\sum_{k=n}^{m} a_k\right| < \dfrac{\epsilon}{3} \\ \left|\dfrac{b_1}{b_n}\displaystyle\sum_{i=1}^{n} a_i\right| < \dfrac{\epsilon}{3} \end{cases}$$

here we do not mention the petty details that how to choose such integer N. Altogether, we get

$$\left|\frac{\sum_{k=1}^{n} a_k b_k}{b_n}\right| < \frac{\epsilon}{3} + \frac{b_n - b_{N+1}}{b_n}\frac{\epsilon}{3} + \frac{\epsilon}{3} = \left(1 + \frac{b_n - b_{N+1}}{b_n} + 1\right)\frac{\epsilon}{3} < \epsilon.$$

125. We obseve that $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$, then it is reminiscent of the formular of summation by parts! Write

$$b_n = (a_1 + 2a_2 + \cdots + na_n)\left(\frac{1}{n} - \frac{1}{n+1}\right),$$

write $s_k = \sum\limits_{i=1}^{k} ia_i$, $b_k = 1/k$, then

$$b_n s_n + \sum_{k=1}^{n-1} b_k = \sum_{k=1}^{n-1} s_k(b_k - b_{k+1}) + b_n s_n = \sum_{k=1}^{n} ka_k b_k = \sum_{k=1}^{n} a_k$$

or

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} a_k - \frac{a_1 + 2a_2 + \cdots + na_n}{n},$$

since we have shown that $\lim\limits_{n\to\infty} \dfrac{a_1 + 2a_2 + \cdots + na_n}{n} = 0$, by the arithmetic rule of operation of limit, we are done.

126. (a) To be added.

(b) To be added.

127. The result directly follows from $\tan^{-1}\dfrac{1}{2n^2} = \tan^{-1}\dfrac{1}{2n-1} - \tan^{-1}\dfrac{1}{2n+1}$, if you really want how to derive this identity, please contact me.

One can also evaluate this by putting $n = 1, 2, \ldots$ and guess the value.

128. Having known the way to derive above identity, then this question is also solved directly.

129. Since

$$\prod_{k=2}^{n+1} \frac{a_k}{a_{k-1}} = (n+1)! \prod_{k=1}^{n} \left(1 + \frac{1}{a_k}\right),$$

we have

$$\prod_{k=1}^{n} \left(1 + \frac{1}{a_k}\right) = \frac{a_{n+1}}{(n+1)!} = \frac{1}{n!} + \frac{a_n}{n!} = \cdots = \frac{1}{n!} + \cdots + \frac{1}{2!} + 1,$$

by the definition of e, we are done.

130. Replacing x by $\frac{x}{2^k}$, we have $T\left(\frac{x}{2^k}\right) - T\left(\frac{x}{2^{k+1}}\right) = b \cdot \frac{x}{2^k} \log \frac{x}{2^k}$. Now taking summation from $k = 0$ to $k = n$, we have $T(x) = T\left(\frac{x}{2^{n+1}}\right) + bx \sum\limits_{k=0}^{n} \frac{1}{2^k} \log \frac{x}{2^k}$, when $n \to \infty$, $T(x) = T(0) + bx \sum\limits_{k=0}^{\infty} \frac{1}{2^k} \log \frac{x}{2^k}$. We are left to evaluate the infinite series whose convergence is a direct consequence of ratio test.

Note that $\log 2x = 2\log x - \log \frac{x}{2}$, we replace x by $\frac{x}{2^{k+1}}$ and divide both sides by 2^k, having

$$\sum_{k=0}^{\infty} \frac{1}{2^k} \log \frac{x}{2^k} = 4 \sum_{k=0}^{\infty} \left(\frac{1}{2^{k+1}} \log \frac{x}{2^{k+1}} - \frac{1}{2^{k+2}} \log \frac{x}{2^{k+2}} \right) = 2 \log \frac{x}{2},$$

hence $T(x) = T(0) + 2bx \log \frac{x}{2}$, the initial condition tells us $T(0)$, yielding

$$T(x) = 1 + 2b \left(\log 2 + x \log \frac{x}{2} \right).$$

131. Observe that $(\log\log k)^{p \log k} = e^{p \log k \log\log\log k} = (k^p)^{\log\log\log k}$, the question can be easily solved. When $p \leq 0$, it diverges obviously. For $p > 0$, since $k > e^{e^{e^{\frac{2}{p}}}} \implies p \log\log\log k > 2$. Take $N = \left[e^{e^{e^{\frac{2}{p}}}} \right] + 1$, $\sum\limits_{N}^{\infty} \dfrac{1}{k^{p \log\log\log k}} < \sum\limits_{k=N}^{\infty} \dfrac{1}{k^2}$, right hand side converges, and so does left hand side.

132. It can be shown by induction that $x_n \in (0, 1)$ and hence $x_{n+1} = x_n(1 - x_n) < x_n$. So $\{x_n\}$ is decreasing and bounded below by 0, $\lim_{n\to\infty} x_n = a \in [0, \frac{1}{4}]$. To show a series diverges, it suffices to compare $\sum x_n$ with other divergent series. For this end, we tend to use comparison test or limit comparison test. Recall that $\sum \frac{1}{n}$ diverges, and the computation of limit $\lim\limits_{n\to\infty} x_n \Big/ \dfrac{1}{n} = \lim\limits_{n\to\infty} n \Big/ \dfrac{1}{x_n} = \lim\limits_{n\to\infty} (1 - x_n) = 1 - a$ is extremely easy by Stolz's theorem. thus the divergence of $\sum x_n$ follows from that of $\sum \frac{1}{n}$.

133. Easy job!

134. To be added.

135. Suppose $b_n < M$, then $\dfrac{1}{a_{n+1}} - \dfrac{1}{a_n} < nM$, it follows that $a_{n+1} > \dfrac{1}{nM + \frac{1}{a_1}}$. Since $\lim\limits_{n\to\infty} \dfrac{1}{n} \Big/ \dfrac{1}{nM + \frac{1}{a_1}} = M \in \mathbb{R}^+$, $\sum_{k=1}^{n} a_k$ diverges, a contradiction.

136. The only nontrivial stuff is convergence of the sequence. It can be seen that since $a_1 = a_2 = 0 \le 1$, so $a_n \le 1$ by induction on n. Since it's bounded above, we hope to prove it's monotonically increasing. Observe that $a_{n+1} - a_n = \frac{2}{3}\left(\frac{a_n^2+b}{2} - a_n\right)$, so $a_{n+1} \ge a_n \iff (a_n - 1)^2 \ge 1 - b \iff 1 - a_n \ge \sqrt{1-b} \iff a_n \le 1 - \sqrt{1-b}$. This is true obviously when $n = 1, 2$. Suppose that's true for $n = k, k+1$, then

$$a_{k+2} = \frac{1}{3}(a_{k+1} + a_k^2 + b) \le \frac{1}{3}(1 - \sqrt{1-b} + (1 - \sqrt{1-b})^2 + b) = 1 - \sqrt{1-b}.$$

The above equality (right one) also reveal that the limit is $1 - \sqrt{1-b}$.

137. Since $a_{k+1} - a_k = \frac{1}{a_k} + \frac{1}{a_{k+1}}$, it is tempting to do a summation $\sum_{k=1}^{n-1}$ on both sides, yielding $a_n - a_1 + \frac{1}{a_1} = \sum_{k=1}^{n-1} \frac{1}{a_k} + \sum_{k=1}^{n} \frac{1}{a_k}$. Let's define $S_n = \sum_{k=1}^{n} \frac{1}{a_k}$, then we get an equality $a_n - a_1 + \frac{1}{a_1} = S_n + S_{n-1}$.

Now by Stolz theorem,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) &= \lim_{n\to\infty} \sqrt{\frac{S_n^2}{n}} \\
&\overset{\text{if RHS exists}}{=\!=\!=} \sqrt{\lim_{n\to\infty} \frac{S_n^2 - s_{n-1}^2}{1}} \\
&= \sqrt{\lim_{n\to\infty} \frac{a_n - a_1 + \frac{1}{a_1}}{a_n}}.
\end{aligned}$$

So it suffices to know the behavier of a_n, it is clear that $\{a_n\}$ is increasing since $a_{n+1} - a_n = \frac{1}{a_{n+1}} + \frac{1}{a_n} > 0$. Then next question is "is it bounded?". Suppose it were bounded, then there would be an M such that $a_n < M$ for all $n \ge 1$, so $a_{k+1} - a_k = \frac{1}{a_{k+1}} + \frac{1}{a_k} > \frac{2}{M}$, but this is already a contradiction since this inequality also implies $a_n - a_1 > \frac{2(n-1)}{M}$, i.e. $\{a_n\}$ unbounded.

We get the limit 1.

138. Assume on the contrary there is $c \in (a, b)$ such that none of subsequence of $\{x_n\}$ can converge to c. Then in particular, there is $\epsilon > 0$ such that $\{x_n : n \in \mathbb{N}\} \cap \big((c - \epsilon, c + \epsilon) \setminus \{c\}\big) = \emptyset$. Since there are at most finitely many $x_n = c$, and hence there is a $K \in \mathbb{N}$,

$$n \ge K \implies x_n \notin (c - \epsilon, c + \epsilon).$$

Let $L = [a, c - \epsilon]$, $R = [c + \epsilon, b]$. Each number in $\{x_n : n \ge K\}$ lies either in L or R.

(i) If there are infinitely many $n \ge K$ such that $x_n \in L \implies x_{n+1} \in R$, then $\{x_n - x_{n+1}\}$ cannot converge to 0, impossible, hence there are finitely many $n \ge K$ such that $x_n \in L \implies x_{n+1} \in R$. In other words, there is an $N \ge K$ such that for all $n \ge N$, $x_n \in L \implies x_{n+1} \in L$.

(ii) Similarly the above argument works when R and L are interchanged, and we can conclude there is $N' \geq K$ such that when $n \geq N'$, $x_n \in R \implies x_{n+1} \in R$.

Now consider two possibilities of $\{x_n\}_{n\geq K}$, firstly, if there are infinitely many x_n's lying on L, then there must be $n' \geq N$, $x_{n'} \in L$, then (i) implies $x_n \in L$ for all $n \geq n'$, in this case $\overline{\lim}_{n\to\infty} x_n = b$ is impossible. Secondly, if there are infinitely many x_n's lying on R, then (ii) implies there is an $n'' \geq N'$ so that $n \geq n'' \implies x_n \in R$, and $\underline{\lim}_{n\to\infty} x_n = a$ becomes impossible.

139. The recursive relation tells us $a_{2n} = a_0 \prod_{j=1}^n \frac{(2j-2)(2j-1)-\ell(\ell+1)}{2j(2j-1)}$, none of $a_n = 0$ (due to the range of ℓ). We expand a bit:

$$a_{2n} = a_0 \prod_{j=1}^n \frac{(2j)(2j-1)-(2(2j-1)+\ell(\ell+1))}{2j(2j-1)} = a_0 \prod_{j=1}^n \left(1 - \underbrace{\left(\frac{1}{j} + \frac{\ell(\ell+1)}{2j(2j-1)}\right)}_{:=b_j}\right) = a_0 \prod_{j=1}^n (1-b_j).$$

As $b_j \to 0$, there is an N such that $j \geq N \implies 1 - b_j > 0$. For $n \geq N$, we have

$$\prod_{j=N}^n (1-b_j) = \exp\left(\ln \prod_{j=N}^n (1-b_j)\right) = \exp\left(\sum_{j=N}^n \ln(1-b_j)\right). \tag{2.3}$$

As $x \to 0$, $\ln(1+x) = x - \frac{1}{2}x^2 + o(x^2)$ (where $o(x^2)/x^2 \to 0$), so there is $\delta > 0$ such that $|x| < \delta \implies |o(x^2)| < x^2$. We may assume N is large enough so that $|b_j| < \delta$ for all $j \geq N$. Thus from (2.3),

$$\begin{aligned}\prod_{j=N}^n (1-b_j) &= \exp\left(\sum_{j=N}^n (-b_j - \tfrac{1}{2}b_j^2 + o(b_j^2))\right)\\ &= \exp\left(\sum_{j=N}^n \left(-\frac{1}{j} - \frac{\ell(\ell+1)}{2j(2j-1)} - \frac{1}{2}b_j^2 + o(b_j^2)\right)\right)\\ &= \exp\left(-\sum_{j=N}^n \frac{1}{j} + A_n\right)\\ &= \exp\left(-\sum_{j=N}^n \frac{1}{j}\right)\exp(A_n),\end{aligned}$$

where $A_n := \sum_{j=N}^n (-\frac{\ell(\ell+1)}{2j(2j-1)} - \frac{1}{2}b_j^2 + o(b_j^2))$. It is clear that A_n is convergent, so e^{A_n} is bounded below by a constant $C > 0$, $n \geq N$. And by considering the graph of $y = \frac{1}{x}$, $\sum_{j=N}^n \frac{1}{j} \leq \int_{N-1}^n \frac{1}{x}\,dx = \ln n - \ln(N-1)$,

$$\prod_{j=N}^n (1-b_j) \geq C \exp\left(-\sum_{j=N}^n \frac{1}{j}\right) \geq Ce^{\ln n^{-1} + \ln(N-1)} = C(N-1)\frac{1}{n}.$$

Finally

$$\begin{aligned}\left|\sum_{n\geq N} a_{2n}\right| &= \left|\sum_{n\geq N} a_0 \prod_{j=1}^{N-1}(1-b_j)\prod_{j=N}^n (1-b_j)\right|\\ &= \left|a_0 \prod_{j=1}^{N-1}(1-b_j)\right| \left|\sum_{n\geq N} \prod_{j=N}^n (1-b_j)\right|\\ &\geq \left|a_0 \prod_{j=1}^{N-1}(1-b_j)\right| \left|\sum_{n\geq N} C(N-1)\frac{1}{n}\right|\\ &= +\infty,\end{aligned}$$

this proves the divergence.

2.5 Binomial Identity

140. By the fact that $\binom{2010}{k} = \frac{2010}{k}\binom{2009}{k-1}$. We subtract right hand side by left hand side,

$$\sum_{k=1}^{2010}\binom{2010}{k}\frac{(-1)^k}{k+2010} - \sum_{k=1}^{2009}\binom{2009}{k}\frac{(-1)^k}{k+2011} = \sum_{k=0}^{2009}\binom{2009}{k}\frac{(-1)^k}{k+2011}\left(\frac{2010}{k+1}+1\right) + \frac{1}{2010}$$
$$= \frac{1}{2010}\left(\sum_{k=0}^{2009}\binom{2010}{k+1}(-1)^{k+1}+1-1\right) + \frac{1}{2010} = 0.$$

141. By infinite binomial series,

$$\begin{aligned}(1-4x)^{-1/2} &= \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-4)^n x^n = \sum_{n=0}^{\infty}\frac{-(1/2)(-3/2)\cdots(-1/2-n+1)}{n!}(-4)^n x^n \\ &= \sum_{n=0}^{\infty} 2^n\frac{(2n-1)!!}{n!}x^n \\ &= \sum_{n=0}^{\infty}\binom{2n}{n}x^n.\end{aligned}$$

142. We use diagonal arrangement to arrange the terms formed by multiplication of two series, i.e.

$$\left(\sum_{i=0}^{n} a_i\right)\left(\sum_{i=0}^{n} b_i\right) = \sum_{k=0}^{n}\left(\sum_{i=0}^{k} a_i b_{n-k}\right).$$

$$\left((1-4x)^{-1/2}\right)^2 = \left(\sum_{n=0}^{\infty}\binom{2n}{n}x^n\right)\left(\sum_{n=0}^{\infty}\binom{2n}{n}x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\right)x^n.$$

On the other hand,

$$\left((1-4x)^{-1/2}\right)^2 = \frac{1}{1-4x} = \sum_{n=0}^{\infty}(4x)^n = \sum_{n=0}^{\infty}2^{2n}x^n,$$

thus, we have

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k} = 2^{2n}.$$

143. To be added.

144. If we do direct expansion, then

$$\sum_{l=1}^{k}(-1)^l\binom{k}{l}a_l = \sum_{l=1}^{k}\sum_{r=1}^{l}(-1)^{l+r}\binom{k}{l}\binom{l}{r}b_r = \sum_{r=1}^{k}\sum_{l=r}^{k}(-1)^{l+r}\binom{k}{l}\binom{l}{r}b_r = \sum_{r=1}^{k-1}\sum_{l=r}^{k}(-1)^{l+r}\binom{k}{l}\binom{l}{r}b_r + b_k.$$

The first term vanishes since

$$\sum_{r=1}^{k-1}\sum_{l=r}^{k}(-1)^{l+r}\binom{k}{l}\binom{l}{r}b_r = \sum_{r=1}^{k-1}\sum_{l=r}^{k}(-1)^{l+r}\binom{k}{r}\binom{k-r}{l-r}b_r = \sum_{r=1}^{k-1}\binom{k}{r}b_r\underbrace{\sum_{l=0}^{k-r}(-1)^l\binom{k-r}{l}}_{=(1+(-1))^{k-r}} = 0.$$

145. Let $a_n = \sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k}\binom{n}{k}$, write it as $a_n = \sum_{k=2}^{n-1}(-1)^{k+1}\frac{1}{k}\binom{n}{k} + n + (-1)^{n+1}\frac{1}{n}$, then combining the identities $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ and $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$, we have $a_n = a_{n-1} + \frac{1}{n}$, so $a_n = \sum_{k=1}^{n}\frac{1}{k}$.

The last equality follows from problem 144.

146. (a) If we count the number of ways to form a group of k people with m of them being leaders, this is easily counted as $\binom{n}{k}\binom{k}{m}$. On the other way round, we first choose people to be leaders, then choose remaining people to form a group of k people, this will be $\binom{n}{m}\binom{n-m}{k-m}$.

(b) To be added.

(c) To be added.

147. To be added.

2.6 Basic Counting

148. We denote 2 disjoint sets respectively A and B. For each number in $[n] = \{1, 2, \ldots, n\}$, they have only 3 destinies. Either to be abandoned, either to be assigned to A or to be assigned to B, hence in general we have $3^n - 1$ choice to put them into A and B. However, none of A and B can be empty, we have $3^n - 1 - (2^n - 1) - (2^n - 1)$. Finally, since grouping C_1 to A and C_2 to B is the same as grouping C_1 to B and C_2 to A, where C_1 and C_2 are two disjoint subsets of $[n]$. We have to divide the number by $2!$, hence the number is $\frac{3^n - 1}{2} - 2^n + 1$.

149. Define s_n be number of way of distribution if exactly n people have present(s), then

$$s_n = (n+1)^6 - \sum_{i=0}^{n-1}\binom{n}{i}s_i,$$

setting $i = 1, 2$ and defining $s_0 = 1$, one can have that

$$\begin{cases} s_0 = 1, \\ s_1 = 2^6 - 1, \\ s_2 = 3^6 - 2\cdot 2^6 + 1, \\ s_3 = 4^6 - 3^7 + 3\cdot 2^6 - 1 = 2100. \end{cases}$$

In case that **all** presents must be distributed, it turns out to be an easy problem. With some basic counting technique, we can have $\frac{6!}{4!} \times 3 + \frac{6!}{2^3} + \frac{6!}{3!2!} \times 3! = 540$.

Alternatively, we define

$$s_n = n^6 - \sum_{k=1}^{n-1}\binom{n}{k}s_k.$$

Setting $n = 1, 2, 3$ to get 3 linear equations, solving them we have $s_3 = 3^6 - 3 - 3(2^6 - 2) = 540$. I think this particular case can help understand how I defined such recurrence relation.

150. We first define a symbol

$$a_n = n \text{ people has at least 1 prize } (n \le 20).$$

Then our desired number is a_5.

To distribute one prize to n people, we have $n+1$ arrangements, that is, either distributing to n people or abandoning this prize, so distributing 20 prize to them can be in $(n+1)^{20}$ ways, including the possibilities that some of them may not have prize.

But everyone must be awarded, we now construct an identity

$$\begin{aligned} a_n &= (n+1)^{20} - N(\text{exactly } n-1 \text{ people have prize}) - N(\text{exactly } n-2 \text{ people have prize}) \\ &\quad - \cdots - N(\text{exactly } 0 \text{ people have prize}) \\ &= (n+1)^{20} - \binom{n}{1} a_{n-1} - \binom{n}{2} a_{n-2} - \cdots - \binom{n}{n} a_0 \end{aligned}$$

with the convention that $a_0 = 1$,

when $n = 1$, $a_1 = 2^{20} - 1$,
when $n = 2$, $a_2 = 3^{20} - 2a_1 - 1$,
when $n = 3$, $a_3 = 4^{20} - 3a_2 - 3a_1 - 1$,

as you can see if we continue this process, since a_5 is expressed in terms of a_1, a_2, a_3, a_4 and these four numbers can be found, we can obtain the expression of a_5 after tedious works,

$$a_5 = 6^{20} - 21 \times 5^{20} + 64 \times 4^{20} - 81 \times 3^{20} - 48 \times 2^{20} - 11.$$

You would find that first 3 questions in this section are intrinsically the same! The number $\frac{s_2}{2}$ is actually the number in the first question. For n different numbers, they are assigned to 2 different groups, giving $s_0 = 1$, $s_1 = 2^n - 1$, $s_2 = 3^n - s_0 - 2s_1 = 3^n - 2^{n+1} + 1$. Since each possible partition is counted twice, we have $\frac{s_2}{2!} = \frac{3^n - 1}{2} - 2^n + 1$. In general, the number $\frac{s_k}{k!}$ is the number of ways to form k groups from n people.

151. Let a_n be such required number. Define $a_n^{(\#)}$ to be the string with length n ending up with a number, we also let $a_n^{(\times)}$ be a string with length n ending up with an operator. Then

$$\begin{aligned} a_n &\triangleq a_n^{(\#)} \\ &= 10 \times a_{n-1}^{(\#)} + 10 \times a_{n-1}^{(\times)} \\ &= 10a_{n-1} + 10(3 \times a_{n-2}^{(\#)}) \qquad \text{(a number must precede or follow the operator)} \\ &= 10a_{n-1} + 30a_{n-2}. \end{aligned}$$

Finally the closed form is a routine calculation, with $a_1 = 10, a_2 = 10 \times 10 = 100$.

152. We first find the probability that $\sum_{i=1}^{6}(a_i - b_i) = 0$. In other words, if we let a_i's be positive and b_i's be negative, the probability is same as finding the constant term in the following expression (0 exponent)

$$\left(\frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6)\right)^6 \left(\frac{1}{6}(x^{-1} + x^{-2} + x^{-3} + x^{-4} + x^{-5} + x^{-6})\right)^6,$$

using the property that some function (e.g. $(1-x)^{-6}$) is analytic for $|x| < 1$, one can find the constant term by special trick. To save time, we use computer to get $\frac{36210119}{544195584}$, so the probability of $\sum_{i=1}^{6} a_i \neq \sum_{i=1}^{6} b_i$ is $1 - \frac{36210119}{544195584} \approx 0.93346$.

As a simple case, we consider the constant term in

$$\left(\frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6)\right)^2 \left(\frac{1}{6}(x^{-1} + x^{-2} + x^{-3} + x^{-4} + x^{-5} + x^{-6})\right)^2,$$

this will be the probability that $a_1 + a_2 = b_1 + b_2$. By direct expansion, we get $\frac{73}{648}$. If we count the probability directly, this will be

$$\begin{aligned}
&\sum_{k=2}^{12} \frac{\#\text{ of possibility that } a_1 + a_2 = k = b_1 + b_2 \text{ (counting permutation)}}{6^4} \\
&= \frac{1}{6^4}(1 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1) \\
&= \frac{73}{648}.
\end{aligned}$$

153. Let the collection of ages of these 10 people be C, define the set

$$S = \{x_1+x_2+\cdots+x_k-\underbrace{(x_{k+1} + \cdots + x_n)}_{n-k\text{ ages}} : x_i \in C, i \neq j \implies x_i \neq x_j, 1 \le k < n, 2 \le n \le 10\}.$$

Our aim is to prove $0 \in S$.

Suppose S does not contain $\{0\}$, then clearly since $\max S \le 52 + 53 + \cdots + 60 - 1 = 503$ and $\min S \ge 1 - (52 + 53 + \cdots + 60) = -503$, $|S| \le 2 \times 503 = 1006$ and

$$x_1 + x_2 + \cdots + x_k \neq x_{k+1} + x_{k+2} + \cdots + x_n, \tag{*}$$

where x_i's are distinct and $1 \le k < n, 2 \le n \le 10$.

Let $x = \min C$, next define

$$\mathcal{A}_j = \big\{x_1 + x_2 + \cdots + x_j - x > 0 : x_i \in C \setminus \{x\}, h \neq k \implies x_h \neq x_k\big\},$$

where $j = 1, 2, \ldots, 9$, then $|\mathcal{A}_j| = \binom{9}{j}$, because if there are

$$\{a_i \in C\backslash\{x\} : i = 1, 2 \ldots, j, h \neq k \implies a_h \neq a_k\} \quad \text{and} \quad \{b_i \in C\backslash\{x\} : i = 1, 2 \ldots, j, h \neq k \implies b_h \neq b_k\}$$

such that

$$a_1 + \cdots + a_j - x = b_1 + \cdots + b_j - x \implies a_1 + \cdots + a_j = b_1 + \cdots + b_j, \tag{**}$$

then (*) tells us $\{a_i : i = 1, \ldots, j\} = \{b_i : i = 1, \ldots, j\}$. Otherwise by cancelling possibly the same term on both sides of (**), we would get a contradiction to (*). That is to say, one combination of j elements in C gives us a unique value.

For $h \neq k$, $\mathcal{A}_h \cap \mathcal{A}_k = \emptyset$ due to the same reason (we don't bother to write down the detail that is messy). We also see that $\mathcal{A}_j \subseteq S$, so the number of "positive" difference is

$$\sum_{k=1}^{9} |\mathcal{A}_k| = \binom{9}{1} + \binom{9}{2} + \cdots + \binom{9}{9} = \sum_{k=0}^{9} \binom{9}{k} - 1 = 2^9 - 1 = 511.$$

Likewise, we have negative difference $-\mathcal{A}_j \triangleq \{-a : a \in \mathcal{A}_j\} \subseteq S$, so $S \supseteq \bigsqcup_{k=1}^{9} \big((-\mathcal{A}_k) \sqcup \mathcal{A}_k\big)$, i.e.

$$1022 = 2 \times 511 = \sum_{k=1}^{9} (|-\mathcal{A}_k| + |\mathcal{A}_k|) \le |S| \le 1006,$$

a contradiction.

154. The total possible outcome is $\dfrac{\binom{6n-1}{2} - 3\big((3n-1) - 1\big) - 1}{3!\binom{6n}{3}} = \dfrac{3n^2 - 3n + 1}{2n\,(3n-1)\,(6n-1)}$.

2.7 Function and Differentiation

155. Assume there is continuous $\gamma : [0,1] \to M$ with $\gamma(0) = (0,0)$ and $\gamma(1) = (1, \sin 1)$. Let $\gamma(t) = (x(t), y(t))$, then continuity of γ implies both x and y are continuous on $[0,1]$. Consider the closed set $x^{-1}(0) := \{t \in [0,1] : x(t) = 0\}$, then $t_0 := \sup x^{-1}(0) < 1$ (since $t_0 \in x^{-1}(0)$).

Now $x(t_0) = 0$ and $x(1) = 1$, and since t_0 is the only point inside $[t_0, 1]$ such that $x(t_0) = 0$, one has by intermediate value theorem,

$$x(t_0, 1] = (0, 1]. \tag{2.4}$$

For each $\alpha \in [0,1]$, there is a sequence $t_n > 0$ such that $t_n \to 0$ and $\sin \frac{1}{t_n} \to \alpha$. By (2.4) for each n there is $s_n > t_0$, so that $x(s_n) = t_n$, and hence

$$\gamma(s_n) = \left(x(s_n), \sin \frac{1}{x(s_n)}\right) = \left(t_n, \sin \frac{1}{t_n}\right).$$

Now $\lim_{n\to\infty} x(s_n) = 0$, but $\{s_n\}$ has a convergent subsequence $\{s_{n_k}\}$, $\lim s_{n_k} = s$ for some $s \geq t_0$. Hence $0 = \lim_{n\to\infty} x(s_n) = \lim x(s_{n_k}) = x(s)$. Since $s \in x^{-1}(0)$, $s \leq t_0$, we conclude $s = t_0$, and $y(t_0) = y(s) = \lim_k y(s_{n_k}) = \lim_{n\to\infty} y(s_n) = \lim_{n\to\infty} \sin \frac{1}{t_n} = \alpha$. But α is arbitrary, a contradiction.

156. Since $[0,1] \times [0,1]$ is compact, f is uniformly continuous on it. Hence for any fixed $\epsilon > 0$ there is δ such that $|x - x'|, |y - y'| < \delta \implies |f(x,y) - f(x',y')| < \epsilon$.

For any x, y, x', y',

$$f(x,y) - g(x') \leq f(x,y) - f(x',y').$$

Hence if we pick $y = y_x$ s.t. $f(x, y_x) = g(x)$, then for all x, x', y',

$$g(x) - g(x') \leq f(x, y_x) - f(x', y').$$

So we can now choose $y' = y_x$,

$$g(x) - g(x') \leq f(x, y_x) - f(x', y_x).$$

Similarly, from $f(x',y') - g(x) \leq f(x',y') - f(x,y)$ for all x, y, x', y', one can choose $y' = y_{x'}$ (as defined above) and $y = y_{x'}$ so that

$$g(x') - g(x) \leq f(x', y_{x'}) - f(x, y_{x'}).$$

These two inequalities say that when $|x - x'| < \delta$, $|g(x) - g(x')| < \epsilon$.

157. (a) To be added

(b) To be added

158. Let K be a compact set such that $f|_{\mathbb{R}\setminus K} \equiv 0$, then the inequality is trivial on $\mathbb{R} \setminus K$ and thus it suffices to show it for $x \in K$. Since we have *finitely* many such a_i's and s_i's, we then invoke the property of compact set that any open cover has *finite* subcover.

Fix an $x \in K$, there must be a $s \in \mathbb{R}$ such that $g(x - s) \neq 0$, and hence there is a $a > 0$ so that $f(x) < ag(x - s)$, which means that

$$\begin{aligned}
x &\in \{x \in \mathbb{R} : f(x) < ag(x-s), \exists a > 0, \exists s \in \mathbb{R}\} \\
&= \bigcup_{a>0} \bigcup_{s\in\mathbb{R}} \{x \in \mathbb{R} : f(x) < ag(x-s)\} \\
&= \bigcup_{(a,s)\in(0,\infty)\times\mathbb{R}} \{x \in \mathbb{R} : 0 < \underbrace{ag(x-s) - f(x)}_{:=F_{a,s}(x)}\} \\
&= \bigcup_{(a,s)\in(0,\infty)\times\mathbb{R}} F_{a,s}^{-1}(0,\infty).
\end{aligned}$$

It is clear that for each a, s, $F_{a,s}$ is continuous and hence $F_{a,s}^{-1}(0,\infty)$ is open. The above inclusion is true for all $x \in K$, we conclude

$$K \subseteq \bigcup_{(a,s)\in(0,\infty)\times\mathbb{R}} \{x \in \mathbb{R} : f(x) < ag(x-s)\},$$

but RHS is an open cover, it can be thinned into a finite subcover, which implies there are $(a_i, s_i) \in (0,\infty)\times\mathbb{R}$, $i = 1, 2, \ldots, n$ such that

$$K \subseteq \bigcup_{i=1}^{n} \{x \in \mathbb{R} : f(x) < a_i g(x - s_i)\}.$$

Finally for each $y \in K$, $y \in \{x \in \mathbb{R} : f(x) < a_j g(x - s_j)\}$, for some $j = 1, 2, \ldots, n$, and hence

$$f(y) < a_j g(y - s_j) \implies f(y) \le \sum_{j=1}^{n} a_j g(y - s_j),$$

and the last inequality is true for all $y \in K$, and we are done.

159. To be added.

160. (a) Direct consequence of mean-value theorem.

(b) Let x_1, x_2 be two consecutive roots of g, the cases $x_1 < x_2 \le 0$ and $0 \le x_1 < x_2$ can be solved by observing that $x_1 g(x_1) = x_2 g(x_2) = 0$ (in that cases there is $c \neq 0$ such that $cf(c) = 0$).

The story is similar in the case $x_1 < 0 < x_2$, there is still $c \in (x_1, x_2)$ such that $cf(c) = 0$. Recall that $xf(x) = \frac{d}{dx}(xg(x)) = xg'(x) + g(x) \implies 0 = cg'(c) + g(c)$, if $c = 0$, then $g(c) = 0$, contradicting that x_1, x_2 are already a pair of consecutive roots. Hence $c \neq 0$ and hence $f(c) = 0$, as desired.

161. Consider the Taylor expansion of f at a,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''\big(\theta(x,a)\big)}{2!}(x-a)^2 \qquad (\heartsuit)$$

where $\theta(x,a)$ is a point lying between x and a. We mainly focus on $x \in [0,1]$ and $a \in (0,1)$.

Taking $x = 0$ and $x = 1$ in ($\heartsuit$), we get

$$0 = f(a) + f'(a)(-a) + \frac{f''\big(\theta(0,a)\big)}{2!}(a)^2, \qquad (1)$$

$$0 = f(a) + f'(a)(1-a) + \frac{f''\big(\theta(1,a)\big)}{2!}(1-a)^2, \qquad (2)$$

next we do the operation $(1)\times(1-a) + (2)\times(a)$,

$$-f(a) = \frac{a(1-a)}{2}\left(f''\big(\theta(0,a)\big)a + f''\big(\theta(1,a)\big)(1-a)\right). \qquad (**)$$

Now we are almost done, take $a \in [0,1]$ such that $f(a) = \sup f([0,1]) = 2$, clearly $a \neq 0, 1$ (i.e. $a \in (0,1)$), so dividing $a(1-a)$ on both sides of (**), we see that

$$\frac{-4}{a(1-a)} = f''\big(\theta(0,a)\big)a + f''\big(\theta(1,a)\big)(1-a),$$

let $c = \min\{f''\big(\theta(0,a)\big), f''\big(\theta(1,a)\big)\}$ and observe a simple fact that $a(1-a) \le \frac{1}{4}$,

$$-16 = \frac{-4}{\frac{1}{4}} \ge \frac{-4}{a(1-a)} = f''\big(\theta(0,a)\big)a + f''\big(\theta(1,a)\big)(1-a) \ge c.$$

162. We observe something simple first. There must be a fixed point of f on $\mathbb{R}$ (we will prove later). That is, there is $x_0 \in \mathbb{R}$ such that $f(x_0) = x_0$, after that taking $x_0 = b = c$, we see that $(x_0, x_0, x_0) \in \mathbb{R}^3$ is indeed a solution.

The solution for such kind of requirement must be unique. If there are two solutions (a_1, b_1, c_1) and (a_2, b_2, c_2), then $a_1 = f \circ f \circ f(a_1)$ and $a_2 = f \circ f \circ f(a_2)$, however, $x - f \circ f \circ f(x)$ is a strictly increasing function (since the monotonicity of x is strict), so the root of $x - f \circ f \circ f(x)$ must be unique, therefore, $a_1 = a_2$. Similarly, $b_1 = b_2$, $c_1 = c_2$, as was to be shown.

So we see the key point of this problem is to show existence of fixed point.

Method 1. Since $x - f(x)$ is strictly increasing with $\lim_{x\to\pm\infty}(x - f(x)) = \pm\infty$, so there is $x_0 \in \mathbb{R}$ such that $x_0 - f(x_0) = 0$.

Method 2. Let's suppose there were no such kind of number. Then either $f(x) > x$ or $f(x) < x$ for all $x \in \mathbb{R}$. Let's assume the former case, then $f(x) < x$, taking $f \circ f$ on both sides,

$$f \circ f \circ f(x) \le f \circ f(x),$$

however, $f(x) < x \implies f \circ f(x) \ge f(x) \overset{x \to f(x)}{\implies} f \circ f \circ f(x) \ge f \circ f(x)$, a contradiction.

The case that $f(x) > x$ for all $x \in \mathbb{R}$ is essentially the same.

163. Let's for the sake of contradiction assume there are just finitely many discontinuous points. It can be seen that discontinuity must occur at the zero of $f(x)$ (otherwise it is not injective), let's call this x_0 (i.e. $f(x_0) = 0$).

There can't be only 1 discontinuous point otherwise $f\big((-\infty, x_0) \cup (x_0, +\infty) \cup \{x_0\}\big) = [-0, \infty) \implies f\big((-\infty, x_0) \cup (x_0, +\infty)\big) = (0, \infty)$, impossible (as left hand side is disconnected). So there have to be at least 2 discontinuous points.

Let $D = \{x_1, x_2, \ldots, x_n\}$ with $x_1 < x_2 < \cdots < x_n$ be the collection of all discontinuous points ($n \ge 2$), then f is unbounded either on the left of x_1 or the right of x_n (this is due to piecewise continuity of f on $[x_1, x_n]$), let's WLOG assume it is unbounded on the right. Then since f is increasing and continuous on $(x_n, +\infty)$, letting $y = \lim_{x\to x_n^+} f(x)$, $f|_{(-\infty, x_n]} : (-\infty, x_n] \to [0, y]$ is bijective.

By "surjective" we mean

$$\begin{aligned} &f\big((-\infty, x_1) \cup (x_1, x_2) \cup \cdots \cup (x_{n-1}, x_n) \cup \{x_1, x_2, \ldots, x_n\}\big) = [0, y] \\ \implies &f\big((-\infty, x_1) \cup (x_1, x_2) \cup \cdots \cup (x_{n-1}, x_n)\big) = [0, y] \setminus \{f(x_1), \ldots, f(x_n)\} \end{aligned}$$

Here $f(x_i) = 0$ and $f(x_j) = y$, for some distinct $x_i, x_j \in D$ (if one of equalities is false, then from the last set equality left hand side is an open set in $\mathbb{R}$ (with the usual topology) while right hand side is not). This implies

$$f\big((-\infty, x_1)\big) \cup f\big((x_1, x_2)\big) \cup \cdots \cup f\big((x_{n-1}, x_n)\big) = (0, y) \setminus \{f(x_{k_1}), \ldots, f(x_{k_{n-2}})\},$$

i.e. disjoint union of n intervals = disjoint union of $n-1$ intervals, impossible. Hence a contradiction arises.

164. Simplifying a little bit, we see that $f(x) + f\left(1 - \frac{1}{x}\right) = 1 + x$, now $x \to 1 - \frac{1}{x}$, we have

$$f\left(1 - \frac{1}{x}\right) + f\left(-\frac{1}{x-1}\right) = 2 - \frac{1}{x},$$

to see what is $f\left(-\frac{1}{x-1}\right)$, we let $x \to -\frac{1}{x-1}$ in the original functional equation, yielding

$$f\left(-\frac{1}{x-1}\right) + f(x) = 1 - \frac{1}{x-1},$$

so

$$1+x-f(x)+1-\frac{1}{x-1}-f(x)=2-\frac{1}{x} \implies f(x)=\frac{x^3-x^2-1}{2x(x-1)}.$$

165. Since X is compact, $f(X)$ is also compact and hence if f^{-1} is continuous on $f(X)$, then f^{-1} must be uniformly continuous. So let's suppose the continuity of f^{-1} is not uniform, then there is $\epsilon > 0$ such that for any $n \in \mathbb{N}$, there are x_n, y_n satisfying $\|x_n - y_n\| < \frac{1}{n}$ and $\|f^{-1}(x_n)-f^{-1}(y_n)\| \geq \epsilon$. Since X is bounded and closed, there are convergent subsequences $\{f^{-1}(x_{n_k})\}$, $\{f^{-1}(y_{n_k})\}$ with respectively the limits x and y. Closedness of X implies $x, y, \in X$ and from the assumption, $\|x-y\| \geq \epsilon$. So

$$x=\lim_{k\to\infty} f^{-1}(x_{n_k}) \implies f(x)=f\left(\lim_{k\to\infty} f^{-1}(x_{n_k})\right)=\lim_{k\to\infty} f\left(f^{-1}(x_{n_k})\right)=\lim_{k\to\infty} x_{n_k}.$$

Similarly, $f(y) = \lim_{k\to\infty} y_{n_k}$, since $\|x_{n_k} - y_{n_k}\| < \frac{1}{n_k}$, $f(y) = f(x)$, hence $x = y$, a contradiction.

166. (a) The uniform continuity implies that $\lim_{x\to 0^+} f(x)$ exists, define the function

$$g(x)=\begin{cases} f(x) & x>0 \\ \lim\limits_{x\to 0^+} f(x) & x=0 \end{cases},$$

we see that g is a uniformly continuous surjective map on $[0,\infty)$.

Suppose that there is a a such that there are only finitely many b's, $f(b) = a$, then there are only finitely many b's. $g(b) = a$. Call the largest one of such b's be b'. Then for all $x > b'$, $g(x) \neq a$, that implies either ①: $g((b,\infty)) = (a,\infty)$ or ②: $g((b,\infty)) = (-\infty, a)$.

Now $g[0,b]$ is bounded and closed, $g([0,\infty)) = g[0,b] \cup g((b,\infty))$, that implies

$$\begin{cases} g([0,\infty)) = g[0,b]\cup(a,\infty) & \text{if ① is true, } g \text{ is bounded below} \\ g([0,\infty)) = g[0,b]\cup(-\infty,a) & \text{if ② is true, } g \text{ is bounded above} \end{cases},$$

both cases lead to contradiction.

(b) $\sqrt{x}\sin\sqrt{x}$ is such an example.

167. We only have 5 possible cases,

(a) $0 < a, b < 1$
(b) $0 < a < 1, b = 1$
(c) $0 < a < 1, b > 1$
(d) $a = 1, b > 1$
(e) $1 < a, b$

Among these cases, only the case $0 < a < 1, b > 1$ is possible, other cases will lead to contradiction. In this case, the identity given is equivalent to $\dfrac{1}{a}-1 = 1-\dfrac{1}{b} \iff ab = \dfrac{a+b}{2}$. We know that $\dfrac{a+b}{2} > \sqrt{ab}$, thus

$$ab > \sqrt{ab} \implies \sqrt{ab}(\sqrt{ab}-1) > 0 \implies ab > 1.$$

168. Since that f is differentiable cannot imply the continuity of f'. Intermediate value theorem fails to work here. We on the contrary define a new continuous function, $g(x) = f(x) - y_0 x$. It can be easily varified that $g'(a) = f'(a) - y_0 < 0$ and $g'(b) = f'(b) - y_0 > 0$, then the minimum value cannot be attained at the end point, the minima must lie somewhere else in the interval (a,b), hence there exists $c \in (a,b)$ such that

$$g'(c) = 0 \text{ (as attained minima)} \implies f'(c) = y_0.$$

169. It is often easy to prove the converse to be impossible for this kind of problems. We on the contrary suppose that $f(x) \neq 0$, $\forall x \in [a, b]$, then either $f(x) > 0$ or $f(x) < 0$ for all $x \in [a, b]$.

We assume that $f(x) > 0$, due to continuity there exists x_0 such that

$$(*) : \qquad f(x) \geq f(x_0) > 0, \forall x \in [a, b]$$

However, we have the following observation.

$$\begin{cases} \exists y_1 \in [a, b], f(y_1) \leq \dfrac{1}{2} f(y_0) \\ \exists y_2 \in [a, b], f(y_2) \leq \dfrac{1}{2} f(y_1) \\ \qquad\qquad \vdots \\ \exists y_n \in [a, b], f(y_n) \leq \dfrac{1}{2} f(y_{n-1}). \end{cases}$$

thus $f(y_n) \leq \frac{1}{2} f(y_{n-1}) \leq \left(\frac{1}{2}\right)^2 f(y_{n-2}) \leq \cdots \leq \left(\frac{1}{2}\right)^n f(y_0)$. Since right hand side tends to 0 as $n \to \infty$, there exists an N such that $n > N \implies f(y_n) < \epsilon < f(x_0)$, a contradiction with (*). The case that $f(x) < 0$ is essentially the same, hence there exists $c \in [a, b]$ such that $f(c) = 0$.

Alternatively, if we know that any bounded sequence have a convergent subsequence, then this problem can be solved in a much neater way. As above we show that

$$|f(y_n)| \leq \frac{1}{2} |f(y_{n-1})| \leq \left(\frac{1}{2}\right)^2 |f(y_{n-2})| \leq \cdots \leq \left(\frac{1}{2}\right)^n |f(y_0)|, \forall n \in \mathbb{N} \cup \{0\},$$

then since there exists a convergent subsequence $\{y_{n_k}\}$ with $\lim\limits_{k\to\infty} y_{n_k} = c \in [a, b]$, thus taking limit on the proved inequality, we get

$$0 \leq \lim_{k\to\infty} |f(y_{n_k})| = \left| f\left(\lim_{k\to\infty} y_{n_k}\right)\right| = |f(c)| \leq \lim_{k\to\infty} \left(\frac{1}{2}\right)^{n_k} |f(y_0)| = 0,$$

here we used the fact that a composite of continuous function is still continuous, we are done.

170. Suppose on the contrary for all $x \in (x_0, x_1)$, $g(x) \neq 0$.

Define

$$G(x) = g(x) f'(x) - f(x) g'(x).$$

The map $x \overset{L}{\mapsto} \frac{1}{x}$ is differentiable, for all $x \neq 0$ and the composition of differentiable functions $L \circ g(x) = \frac{1}{g(x)}$ is differentiable, moreover, the multiplication of two differentiable functions $f(x) \cdot \frac{1}{g(x)}$ is also differentiable, that means the first order derivative of $\frac{f(x)}{g(x)}$ exists and its formula is given by

$$\frac{G(x)}{g(x)^2} = \frac{g(x) f'(x) - f(x) g'(x)}{g(x)^2} = \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right).$$

Now since $f(x)/g(x)$ is continuous on $[x_0, x_1]$, differentiable on (x_0, x_1) and $f(x_0)/g(x_0) = f(x_1)/g(x_1) = 0$, by Rolle's theorem, there exists a $c' \in (x_0, x_1)$ such that

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\bigg|_{x=c'} = \frac{G(c')}{g(c')^2} = 0 \implies G(c') = 0,$$

a contradiction with $G(x) \neq 0$ for all $x \in (a, b)$. Thus the assumption is wrong and there exists a $c \in (x_0, x_1)$ such that $g(c) = 0$.

171. Suppose $f''(x) \neq 0$ for all real x, then we have either $f'' > 0$ or $f'' < 0$ for all x.

Case that $f'' > 0$:

$f'' > 0 \implies f$ is convex on $\mathbb{R}$. Let $u < x < v < w$, then we have

$$\frac{f(x)-f(u)}{x-u} < \frac{f(v)-f(x)}{v-x} \iff \boxed{f(u) > f(x) + \frac{u-x}{v-x}(f(v)-f(x))} \tag{1}$$

$$\frac{f(v)-f(x)}{v-x} < \frac{f(w)-f(v)}{w-v} \iff \boxed{f(w) > f(v) + \frac{w-v}{v-x}(f(v)-f(x))}. \tag{2}$$

Case 1. If there exist x and v such that $f(v)-f(x) < 0$, then by using inequality (1), we have as $u \to -\infty$, $f(u) \to +\infty$.

Case 2. If there exist x and v such that $f(v)-f(x) > 0$, then by using inequality (2), we have as $w \to +\infty$, $f(w) \to +\infty$.

Case 3. Unfortunately if $f(x) = f(v)$, then take a suitable value v^*, $u < x < v < v^* < w$, such that $f(x) \neq f(v^*)$, same conclusion as above.

Hence a function which is convex on the whole real line must be unbounded on $\mathbb{R}$, a contradiction with boundedness of f.

Case that $f'' < 0$:

Let $g = -f$, then g is a convex function on $\mathbb{R}$. We have proved that a convex function is unbounded, hence g is unbounded, and hence f is also unbounded, again a contradiction with boundedness of f.

172. To be added.

173. Let $F(x) = \int_0^x f(t)\,dt$, consider Taylor expansion about α, then

$$F(x) = F(\alpha) + f(\alpha)(x-\alpha) + \frac{f'(c)}{2}(x-\alpha)^2, \tag{*}$$

for some c lies between x and α. Since $F(0) = F(1) = 0$, let $x = 0$ and $x = 1$ in (*), denoting respectively the constant c by c_0 and c_1 respectively, we get

$$F(\alpha) + f(\alpha)(-\alpha) + \frac{f'(c_0)}{2}(\alpha)^2 = 0 = F(\alpha) + f(\alpha)(1-\alpha) + \frac{f'(c_1)}{2}(1-\alpha)^2$$

getting rid of the term $f(\alpha)$, we get

$$\begin{aligned}
\left|\int_0^\alpha f(x)\,dx\right| = |F(\alpha)|(\alpha + (1-\alpha)) &= \left|\frac{f'(c_0)}{2}\alpha^2(1-\alpha) + \frac{f'(c_1)}{2}(1-\alpha)^2\alpha\right| \\
&\le \frac{\max_{0\le x\le 1}|f'(x)|}{2}|(1-\alpha)(\alpha)| \\
&\le \frac{\max_{0\le x\le 1}|f'(x)|}{2}\cdot\left|\frac{1}{4}\right| \\
&= \frac{1}{8}\max_{0\le x\le 1}|f'(x)|.
\end{aligned}$$

174. Just make good use of the expansion $f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{3!}f'''(x+\theta h)h^3$, for some $\theta \in (0,1)$.

175. For the first fact, we again use the taylor expansion $f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x+\theta h)h^2$, for some $\theta \in (0,1)$, replace h by $-h$ to construct another equation (remember to choose different θ's), subtract two equations, observe that $0 \le 2M_0 + 2M_1h + M_2h^2$ for any h, while discriminant ≤ 0, we are done.

For the second fact, in exactly the same manner we conclude that $M_{j+1} \le \sqrt{2M_jM_{j+2}}$ for all $j \le p-2$. Now we take product $\prod_{j=m}^{n}$ on both sides, having

$$\sqrt{M_{n+1}M_{m+1}} \le (\sqrt{2})^{n-m+1}\sqrt{M_{n+2}M_m}. \tag{*}$$

Before we proceed, we first consider two cases. If $M_k = 0$, then the inequality we are asked to prove obviously holds since right hand side is always non-negative. In case if $M_k > 0$, then we take the product $\prod_{m=0}^{k-1}$ on both sides of (*), it results in

$$M_n^k \le \left(\frac{M_0}{M_k}2^{k(2n-k+1)/2}\right)M_{n+1}^k.$$

(n is repalced by $n-1$ for making the inequality seem better) We are interested in this because it is a beautiful (in the sense of solving the problem) recurrence relation, we have a direct consequence

$$\begin{aligned} M_n^k &\le \left(\frac{M_0}{M_k}\right)^{p-k} 2^{\sum_{j=n}^{n+p-k-1}k(2j-k+1)/2}M_{n+p-k}^k \\ &= \left(\frac{M_0}{M_k}\right)^{p-k} 2^{k(p-k)(2n-2k+p)/2}M_{n+p-k}^k. \end{aligned}$$

Finally, we take $n = k$, $M_k^p \le M_0^{p-k}2^{\frac{k(p-k)p}{2}}M_p^k$, done.

Remark. *We have to take care that $k-1$ is the value that m can take, recall that at the beginning we introduce the product $\prod_{j=m}^{n}$, that means $m \le n$, so $k-1 \le n \iff k \le n+1$, hence the choice $k = n$ is possible.*

176. $f'(0) = m \iff \lim_{x\to 0}\frac{f(x)-f(0)-mx}{x} = 0$. We define $g(x) = f(x) - f(0) - mx$, then given equality can be simplified to

$$\lim_{x\to 0}\frac{g(2x)-g(x)}{x} = 0.$$

Our next target is to show $\lim_{x\to 0} g(x)/x = 0$, this is left as exercise.

177. Observe that $|f - f'| = e^x\left|\frac{d}{dx}(e^{-x}f)\right|$, to tackle the problem, we have something to work on the function $e^{-x}f$, define $g(x) = e^{-x}f(x)$, then

$$g'(t) = e^{-t}\big(f'(t) - f(t)\big) \le e^{-t}|f'(t) - f(t)| \le e^{-t},$$

it follows that

$$\int_0^x g'(t)\,dt \le \int_0^x e^{-t}\,dt \implies f(x) \le -(1+e^{-x})e^x < -e^x,$$

this shows that $\lim_{x\to\infty} f(x) = -\infty$.

178. It is easy to prove that

$$f(x_0) = x_0, \exists x_0 \in [0,1]. \tag{*}$$

Suppose for all $x \in [0,1]$, $h(x) = f(x) - g(x) \ne 0$, then either ①: $\boxed{h(x) > 0}$ or ②: $\boxed{h(x) < 0}$ for all $x \in [0,1]$.

For case ①, It follows from (*) that $g\big(f(x_0)\big) = g(x_0) = f\big(g(x_0)\big)$, that implies if x_0 is a solution of

$$f(x) = x, \tag{**}$$

then $g(x_0)$ is also a solution of $f(x) = x$, moreover, we assume that ① holds, that means $x_0 = f(x_0) > g(x_0)$. If $x_0 = 0$, then $0 = x_0 > g(x_0)$, a contradiction.

Suppose now $x_0 > 0$, again, since $x_1 = g(x_0)$ is a solution, $x_2 = g(x_1)$ is also a solution, while $x_1 = f(x_1) > g(x_1) = x_2$, it tells us we can inductively define a sequence $x_{n+1} = g(x_n)$ that satisfies

$$\begin{cases} f(x_n) = x_n \\ x_n > x_{n+1} \text{ (due to the fact that } f > g) \\ x_n > 0 \text{ (if one of them is zero, then we get } 0 = x_k > x_{k+1}\text{, impossible).} \end{cases}$$

Since $\{x_n\}$ is decreasing and bounded below, it is convergent with limit $a \in [0, 1)$. Now there exists $u \in [0, 1]$ such that $h(x) \geq h(u) > 0$, by taking $x = x_n$, we get for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\begin{aligned} n > N \implies 0 < h(u) \leq h(x_n) &= |x_n - x_{n+1}| \\ &\leq |x_n - a| + |x_{n+1} - a| < \epsilon. \end{aligned}$$

That means a fixed positive number can be arbitrarily small, a contradiction. The case ② also leads to a similar contradiction, we are done.

179. If $f'(0)f'(1) \leq 0$, then from mean value theorem,

$$|f'(0)| + |f'(1)| = |f'(0) - f'(1)| = |f''(l)||0 - 1| \leq 2010.$$

Suppose now $f'(0)f'(1) > 0$, say $f'(0), f'(1) > 0$, then by Taylor's expansion with remainder term,

$$\begin{cases} f'(x) = f'(0) + f''(c_1)x & \ldots (1) \\ f'(x) = f'(1) + f''(c_2)(x - 1) & \ldots (2) \end{cases}$$

adding (1) and (2), we have

$$f'(0) + f'(1) + f''(c_1)x + f''(c_2)(x - 1) = 2f'(x). \tag{*}$$

It is given that there exists a $c \in (0, 1)$ such that $f(c) > f(1)$, it follows that $f'(L) \overset{\exists L \in (c,1)}{=\!=\!=} \frac{f(c)-f(1)}{c-1} < 0$, let $x = L$ in (*) such that its right hand side becomes less than 0, we have

$$0 \leq f'(0) + f'(1) \leq -f''(c_1)L + f''(c_2)(1 - L) \leq 2010(L) + 2010(1 - L) = 2010.$$

Similarly, when both $f'(0), f'(1) < 0$, by the fact that there exists a $c \in (0, 1)$ such that $f'(K) = \frac{f(c)-f(0)}{c-0} > 0$, we can put $x = K$ in (*) to make right hand side become positive, this yields

$$0 < -f'(0) - f'(1) \leq f''(c_1')K + f''(c_2')(K-1) \leq |f''(c_1')|K + |f''(c_2')||K-1| \leq 2010(K+1-K) = 2010.$$

180. We see that $wp(w) = 1$, letting $g(x) = xp(x) - 1$, we have $g(x) = 0$ for $x = 1, 2, \ldots, 2^n$. As $g(x)$ is a polynomial of degree $n + 1$ possessing $1, 2, \ldots, 2^n$ as its roots, for some non-zero constant C, we have

$$g(x) = C\prod_{k=0}^{n}(x - 2^k) = xp(x) - 1 \iff p(x) = \frac{\overbrace{C\prod_{k=0}^{n}(x - 2^k) + 1}_{\text{Call this } H(x).}}{x}.$$

Now $H(0) = C\prod_{k=0}^{n}(-2^k) + 1 = C(-1)^{n+1}\prod_{k=0}^{n} 2^k + 1 = C(-1)^{n+1}2^{n(n+1)/2} + 1$. As $p(x)$ is a polynomial, its constant term must be zero, hence $H(0) = 0$, this implies

$$C(-1)^{n+1}2^{n(n+1)/2} + 1 = 0 \iff C = (-1)^n 2^{-n(n+1)/2}.$$

As $p(x)$ is continuous everywhere, $p(0) = p(\lim_{x\to 0} x) = \lim_{x\to 0} p(x)$, but the limit $\lim_{x\to 0}\frac{H(x)}{x}$ now is of $\frac{0}{0}$ form, we apply L'hôspital's rule once, this yields

$$\begin{aligned} p(0) &= \lim_{x\to 0} H'(x) = C\lim_{x\to 0}\frac{d}{dx}\left(\prod_{k=0}^{n}(x-2^k)\right) \\ &= C\sum_{p=0}^{n}\prod_{\substack{k=0\\k\neq p}}^{n}(-2^k) = C\left((-1)^n\sum_{p=0}^{n}2^{\sum_{k=0}^{n}k}2^{-p}\right) \\ &= \left((-1)^n 2^{-n(n+1)/2}\right)\left((-1)^n 2^{n(n+1)/2}\sum_{p=0}^{n}2^{-p}\right) \\ &= \sum_{p=0}^{n}2^{-p} = 2 - 2^{-n}. \end{aligned}$$

181.

- Let $y = 0$, we have $0 \le f(0)$.
- While taking $y = -x$, we have $f(0) \le f(x) + f(-x) \implies f(0) \le x\left(\frac{f(x)}{x} - \frac{f(-x)}{(-x)}\right)$ for all $x \setminus \{0\}$, so $x \to 0 \implies f(0) \le 0$ (for otherwise, $f(0) > 0$ implies $0 < f(0) \le 0$ when $x \to 0$, a contradiction).
- Altogether we get $f(0) = 0$, this also implies from the above inequality,

$$f(x) \ge -f(-x). \tag{*}$$

- Observe that $f(2^n x) \le 2^n f(x)$, we get $f(x) \le x\left(\frac{f(\frac{x}{2^n})}{\frac{x}{2^n}}\right)$, taking $n \to \infty$, we get $f(x) \le cx$, for all $x \in \mathbb{R}\setminus\{0\}$. But in addition, $f(0) = 0$, so for all $x \in \mathbb{R}$,

$$f(x) \le cx. \tag{**}$$

- From (*) and (**), $f(x) \ge -f(-x) \ge cx$, we have $f(x) = cx$.

182.

- By taking everything zero, we get $f \circ f(0) = 0$.
- If $x = 0$, then $f \circ f(y) = f(0) - y$. (*)
- Let $x = f(0), y = 0$, then $f\big(2f(0)\big) = f(0)$, by using the identity just above,

$$f(0) - 2f(0) = f\left(f\big(2f(0)\big)\right) = f\big(f(0)\big) \implies -f(0) = f(0) \implies f(0) = 0.$$

  Now (*) becomes $f \circ f(y) = -y$. (**)
  Taking f on (**) once, we see that $-y = f(-y)$, i.e. f is odd.
- Replacing y by $-f(y)$,

$$f(x+y) = f(x) + f(y),$$

  so f satisfies Cauchy functional equation, in particular,

$$f(x) = f(1)x, \forall x \in \mathbb{Z}.$$

  Finally, by taking $x = f(1)$, we get from (**) that $f(1)^2 = -1$, a contradiction.

183. What we are asked to prove is the same as proving there is a a interval $(-\delta', \delta')$ such that $f(x) \geq f(0) = 0$.

Case 1. Suppose that $x \geq 0$, then consider Taylor's expansion about 0,

$$g(x) = g'(0)x + g''(h_x)\frac{x^2}{2}, \text{ for some } h_x \text{ lies between 0 and } x. \qquad (*)$$

Now there is M such that $|g''| \leq M$ for all $x \in J$, so (*) tells us

$$g(x) \geq g'(0)x - M\frac{x^2}{2}.$$

Now choose $x > 0$ such that $g'(0)x - M\frac{x^2}{2} \geq 0 \iff \delta = \frac{2g'(0)}{M} \geq x$. While since clearly when $x = 0$ the inequality still holds, altogether

$$g(x) \geq 0, \forall x \in [0, \delta).$$

Let's replaced x by x^2t, but we still need $x^2t < \delta$. Since $t \leq x$ in the integration domain, if we choose $x < \delta^{1/3}$, then $x^2t \leq x^3 < \delta$, so when $x \in [0, \delta^{1/3})$,

$$g(x^2t) \geq 0 \implies f(x) = \int_0^x g(x^2t)\,dt \geq 0 = f(0).$$

Case 2. Suppose now $x < 0$, let $u > 0$ and let's replace x by $-u$, then we need to show

$$f(-u) = \int_0^{-u} g(u^2t)\,dt \geq 0.$$

This is quite clear since $f(-u) = -\int_{-u}^0 g(u^2t)\,dt$, but $g(u^2t) \leq 0$ for sufficiently small $u > 0$. It remains to find such u (easy from (*) again). And this is true when $u \in (0, \delta^{1/3})$, i.e. $x \in (-\delta^{1/3}, 0)$, so $f(x) \geq 0$ when $x \in (-\delta^{1/3}, \delta^{1/3})$.

184. Since $f(r + 1/n) = f(r)$ and $f(r - 1/n) = f(r)$, for all $r \in \mathbb{Q}$ and $n \in \mathbb{N}$, so

$$f\left(r + \frac{m}{n}\right) = f\left(r + \frac{m-1}{n}\right) = \cdots = f(r)$$

and

$$f\left(r - \frac{m}{n}\right) = f\left(r - \frac{m-1}{n}\right) = \cdots = f(r),$$

for all $r \in \mathbb{Q}$ and $m, n \in \mathbb{N}$, so $f(y + x) = f(y)$, for all $x, y \in \mathbb{Q}$. As a result, $f(x) = f(0)$, for all $x \in \mathbb{Q}$. As f is continuous, $f(x) = f(0)$, for all $x \in \mathbb{R}$.

185. **Method 1.** Observe that

$$\begin{aligned}|f(x)| &\leq \int_0^x f(x_1)\,dx_1 \leq \int_0^x |f(x_1)|\,dx_1 \leq \int_0^x \int_0^{x_1} f(x_2)\,dx_2dx_1 \\ &\leq \cdots \leq \int_0^x \int_0^{x_1} \cdots \int_0^{x_{n-1}} |f(x_n)|\,dx_ndx_{n-1}\cdots dx_1. \qquad (*)\end{aligned}$$

Since $|f(x)|$ is continuous whose domain $[0, 1]$ is compact (closed and bounded on $\mathbb{R}$), so $\sup_{x\in[0,1]} |f(x)| < +\infty$, hence from (*),

$$|f(x)| \leq x^n \sup_{x\in[0,1]} |f(x)|,$$

so for each $x \in [0,1)$, f is forced to be zero by letting $n \to \infty$. Due to continuity, $f(1) = 0$ since

$$f(1) = f\left(\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)\right) = \lim_{n\to\infty} f\left(1 - \frac{1}{n}\right) = 0.$$

Method 2. Suppose $f(a) \neq 0$, for some $a \in (0,1]$, we choose $x_0 \in [0,a]$ such that $|f(x_0)| = \sup_{x\in[0,a]} |f(x)|$. For $x \in [0,a]$,

$$|f(x)| \leq \int_0^x |f(t)|\, dt \leq |f(x_0)|x \leq |f(x_0)|a,$$

putting $x = x_0$ in above inequality, we deduce that $a \geq 1$, but $a \leq 1$, so $a = 1$.

Hence by contrapositive, if $x \neq 1$, $f(x) = 0$, once again due to continuity $f(1) = 0$, a contradiction.

186. WLOG we assume the leading coefficient of P is positive.

When $r = 0$, we are done. Suppose $r \neq 0$, we write

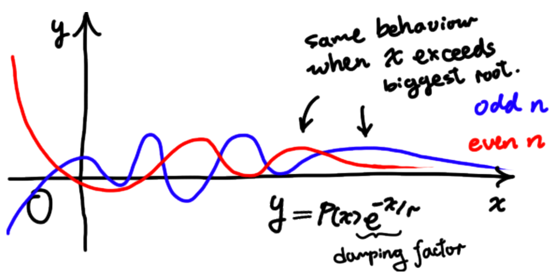
$$Q_r(x) = -re^{x/r}(P(x)e^{-x/r})'. \tag{*}$$

The proof constitutes 3 claims.

- We always get a biggest root of $Q_r(x)$ that is larger than biggest root of $P(x)$.
- Between two nearest distinct roots h, k $(h < k)$, there is $c \in (h,k)$ so that $(P(x)e^{-x/r})'|_{x=c} = 0$.
- Root with multiplicity e of P will give root with multiplicity **at least** $e - 1$ of Q_r.

The 3rd claim is easy, 2nd claim is direct consequence of mean value theorem.

Proof of 1st claim. From the graph it is intuitive that after the line $(x, P(x)e^{-x/r})$ passes through the biggest root, we always get a point such that $(P(x)e^{-x/r})'$ vanishes.



Let's make it precise. Let α be the biggest root of $P(x)$. If $(P(x)e^{-x/r})' \neq 0$, for all $x > \alpha$, then due to continuity $(P(x)e^{-x/r})' > 0$ or $(P(x)e^{-x/r})' < 0$, for all $x > \alpha$. The former case implies $\lim_{x\to+\infty} P(x)e^{-x/r} \neq 0$, the latter case implies $P(x)e^{-x/r} < 0$ for all $x > \alpha$, both are contradictions. So there is $\alpha' > \alpha$ such that $(P(x)e^{-x/r})'|_{x=\alpha'} = 0$, i.e. $Q_r(\alpha') = 0$. $\square$

We can now solve the problem. Let $P = A(x-\alpha_1)^{e_1}(x-\alpha_2)^{e_2}\cdots(x-\alpha_r)^{e_r}$, α_i's are distinct, $\sum_{i=1}^r e_i = n$ and $e_i \geq 1$. WLOG, suppose that $e_1, \ldots, e_k \geq 2$ and $e_{k+1}, \cdots, e_r = 1$, then Q_r can be factorized into a product of linear factors with degree at least

$$\underbrace{\sum_{i=1}^k (e_i - 1)}_{\text{due to multiplicity}} + \underbrace{r-1}_{r \text{ distinct roots}} + \underbrace{1}_{\text{largest root}} = \sum_{i=1}^k e_i + (r-k) = n.$$

So we get all the roots we need, they are indeed real.

187. **Method 1.** We show by sequential continuity theorem. Suppose there is $x_n \to a$, due to differentiability, for any $\epsilon > 0$, there is always a $y_n \in \big((x_n - \frac{1}{n}, x_n + \frac{1}{n}) \setminus \{x_n\}\big) \cap I$, such that

$$\left| f'(x_n) - \frac{f(y_n) - f(x_n)}{y_n - x_n} \right| < \frac{\epsilon}{2},$$

now the following estimate will do

$$|f'(x_n) - f'(a)| \le \left| f'(x_n) - \frac{f(y_n) - f(x_n)}{y_n - x_n} \right| + \left| \frac{f(y_n) - f(x_n)}{y_n - x_n} - f'(a) \right|$$

since with the chosen ϵ above, we can find an N such that when $n > N$, $|f'(x_n) - f'(a)| < \epsilon$.

Method 2. We imitate the proof of sequential continuity theorem! Suppose f' is not continuous at a, then there is $\epsilon > 0$ such that for each $n \in \mathbb{N}$, there is $x_n \in \big(a - \frac{1}{n}, a + \frac{1}{n}\big) \cap I$, $|f'(x_n) - f'(a)| \ge \epsilon$. Define

$$g(x) = \begin{cases} \left| \dfrac{f(x_n) - f(x)}{x_n - x} - f'(a) \right|, & x \ne x_n, \\ |f'(x_n) - f'(a)|, & x = x_n, \end{cases}$$

it is obviously continuous when $x \ne x_n$, and it is also continuous when $x = x_n$ due to differentiability at this point. Since $g(x_n) \ge \epsilon > \epsilon/2$, so there is $\delta_n > 0$ such that when $y \in (x_n - \delta_n, x_n + \delta_n) \cap I$, we still have $g(y) > \epsilon/2$.

Let $\{K_n \in \mathbb{N}\}$ be strictly increasing such that $\frac{1}{K_n} < \delta_n$, then if we take $y = y_n \in (x_n - \frac{1}{K_n}, x_n + \frac{1}{K_n}) \cap I \setminus \{x_n\}$, we also have $g(y_n) > \epsilon/2$, or $\left| \dfrac{f(x_n) - f(y_n)}{x_n - y_n} - f'(a) \right| > \epsilon/2$, for all $n \ge 1$, this is a contradiction as both $x_n, y_n \to a$, $x_n \ne y_n$.

188. (a) Let's assume on the contrary there is $x \in (a, b)$ such that $f(x) > f(b)$. Let $x_0 \in [x, b]$ such that $f(x_0) = \max f([x, b])$. Then for sure

$$f(x_0) \ge f(x) > f(b),$$

but that means $x_0 \ne b$, hence $x_0 \in [x, b)$. But then since x_0 is a shadow point, there is $x' > x_0$ so that $f(x') > f(x_0)$. If $x' > b$ then b becomes a shadow point, that is impossible. So we know that $x' \in (x_0, b] \subseteq (x, b]$. But we have $f(x_0) \ge f(x') > f(x_0)$, a contradiction.

(b) If $f(a) < f(b)$, then a is a shadow point, not allowed, so $f(a) \ge f(b)$. Now for each $x \in (a, b)$, $f(x) \le f(b) \le f(a)$, so $f(a) = \lim_{x \to a^+} f(x) \le f(b) \le f(a)$ implies $f(a) = f(b)$.

189. Let $a_k = \sqrt{2k\pi}, b_k = \sqrt{2k\pi + \frac{\pi}{4}}$, then we repeatedly use mean value theorem on the interval $[a_k, b_k]$ to get

$$f(b_k) - \sin b_k^2 - (f(a_k) - \sin a_k^2) = f'(c_k) - 2c_k \cos c_k^2,$$

where $c_k \in (a_k, b_k)$. We rearrange the terms to get $f'(c_k) = f(b_k) - f(a_k) - \sin b_k^2 + \sin a_k^2 + 2c_k \cos c_k^2$. Since $c_k \in (a_k, b_k)$, hence $c_k^2 \in (2k\pi, 2k\pi + \frac{\pi}{4})$ and $\cos c_k^2 \ge \frac{1}{\sqrt{2}}$. Moreover, $|f(b_k)| \le \frac{1}{4} + |\sin x^2| \le \frac{5}{4}$, we get

$$f'(c_k) \ge 2c_k \cdot \frac{1}{\sqrt{2}} - \frac{5}{4} \times 2 - 1 \times 2 = c_k\sqrt{2} - \frac{9}{2},$$

hence $\lim_{k \to \infty} f'(c_k) = +\infty$.

190. To be added.

191. To be added.

192. Let $Z(f) = f^{-1}(0)$, we will prove that f is holomorphic on $\mathbb{C} \setminus Z(f)$ and $Z(f)$ respectively, let $z_0 \notin Z(f)$, then there is $\delta > 0$ such that $f(z) \neq 0$ on $B(z_0, \delta)$, since $B(z_0, \delta)$ is simply connected and doesn't contain zero of f, we can define a holomorphic branch of $\log f^2$ on $B(z_0, \delta)$ by

$$\log f^2(z) = \int_{z_0}^{z} \frac{(f^2)'}{f^2}\, dz + \log f^2(z_0)$$

with fixing a choice of $\log f^2(z_0)$. Now $g = \frac{1}{2}\log f^2$ is also holomorphic, so is $h = e^g$. We notice that h is a holomorphic square root of f^2, that is, $h^2 = f^2 \implies h(z_0) = \pm f(z_0)$.

If $h(z_0) = f(z_0)$, we hope that in a neighborhood of z_0, $h(z) = f(z)$, this is indeed true since

$$0 = f^2 - h^2 = (f + h)(f - h),$$

as $h(z_0) + f(z_0) \neq 0$ (we assumed $z_0 \notin Z(f)$ at the beginning), so near z_0, $h(z) + f(z) \neq 0$ $\implies$ near z_0, $f(z) - h(z) = 0 \implies f$ is holomorphic near z_0.

The case that $h(z_0) = -f(z_0)$ is essentially the same.

As z_0 is arbitrary in $G \setminus Z(f)$, we conclude that f is holomorphic away from its root.

By considering the restriction of f to $G \setminus Z(f)$, then $f|_{G\setminus Z(f)}$ is a holomorphic function with singularity on $Z(f)$, since f is continuous, for each $z_1 \in Z(f)$, $\lim_{z\to z_1} f(z) \in \mathbb{C}$, hence z_1 is a removable singularity, and hence f is holomorphic on $Z(f)$, we are done.

Remark. *We can also argue $f = h$ near z_0 by seeing that*

for any neighborhood of z_0, there is z in that neighborhood such that $h(z) \neq f(z)$,

is wrong.

193. Method 1. We claim that $f^{(n)}(x) \in \mathbb{R}$ for any $x \in \mathbb{R}$. Let $x \in \mathbb{R}$. We first prove that $f'(x)$ is real, this is because $f'(x) = \lim_{\mathbb{R}\ni h\to 0} \frac{f(x+h)-f(x)}{h} \in \mathbb{R}$. Suppose $f^{(n-1)}(x)$ is real for every $x \in \mathbb{R}$, then $f^{(n)}(x) = \lim_{\mathbb{R}\ni h\to 0} \frac{f^{(n-1)}(x+h)-f^{(n-1)}(x)}{h} \in \mathbb{R}$ due to the same reason, so we have proved our claim. In particular,

$$f^{(2n)}(0) \in \mathbb{R}, \forall n \in \mathbb{N}. \tag{*}$$

Let $g(z) = f(z) + f(-z)$, then since f is entire, the power series representation $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ holds for all $z \in \mathbb{C}$. So $g(z) = \sum_{n=0}^{\infty} \frac{2f^{(2n)}(0)}{(2n)!} z^{2n}$. Let $y \in \mathbb{R}$, consider $z = iy$, from (*),

$$i\mathbb{R} \ni f(iy) + f(i(-y)) = g(iy) = \sum_{n=0}^{\infty} \frac{2f^{(2n)}(0)}{(2n)!}(-1)^n y^{2n} \in \mathbb{R},$$

this can only happen when $g(iy) = 0$. Replace y by $\frac{1}{n}$ with large enough $n \in \mathbb{N}$, then $g(i\frac{1}{n}) = 0$ and $\frac{1}{n} \to 0$, so $g(z) = 0$ for all z.

Method 2. Alternatively, since an entire function which takes real value on the real axis must satisfy $f(z) = \overline{f(\overline{z})}$ by Schwarz reflection principle, so by taking $z = i\frac{1}{n}$, $n \in \mathbb{N}$, we are done.

194. To be added.

195. Since $f(z)e^{i\alpha} + g(z)e^{i\beta}$ as a function in z is holomorphic on $\overline{B(0,1)}$ (in particular, continuous on the boundary), for any fixed α, β, there is $w_{\alpha,\beta} \in \partial B(0,1)$ such that

$$\begin{aligned} |f(z)e^{i\alpha} + g(z)e^{i\beta}| &\leq |f(w_{\alpha,\beta})e^{i\alpha} + g(w_{\alpha,\beta})e^{i\beta}| \leq |f(w_{\alpha,\beta})| + |g(w_{\alpha,\beta})| \\ &\leq \sup_{w\in\partial B(0,1)} (|f| + |g|)(w), \end{aligned}$$

for all $z \in \overline{B(0,1)}$. By the compactness of $\partial B(0,1)$, $\sup_{w\in\partial B(0,1)}(|f|+|g|)(w) = (|f|+|g|)(z_0)$ for some $z_0 \in \partial B(0,1)$, so the last inequality implies for all $z \in \overline{B(0,1)}$,

$$|f(z)e^{i\alpha} + g(z)e^{i\beta}| \le (|f| + |g|)(z_0).$$

However, for each $z \in \overline{B(0,1)}$, there is α_z, β_z such that $f(z)e^{i\alpha_z} = |f(z)|$ and $g(z)e^{i\beta_z} = |g(z)|$, so the last inequality implies for all $z \in \overline{B(0,1)}$,

$$|f(z)| + |g(z)| \le (|f| + |g|)(z_0),$$

and the upper bound is attained when $z = z_0$.

196. Let $H = \{z : \operatorname{Re} z > 0\}$, the right half space. Then we can define a conformal map $T : B(0,1) \to H$ by $T(z) = \frac{1-z}{1+z}$. Then $g \triangleq T^{-1} \circ f \circ T : B(0,1) \to B(0,1)$ satisfies $g(0) = T^{-1}\big(f(T(0))\big) = T^{-1}(f(1)) = T^{-1}(1) = 0$ and holomorphic on $B(0,1)$. By Schwarz's lemma, one has $|g(z)| \le |z|$. By putting $z = -\frac{1}{3}$,

$$\left|\frac{1-f(2)}{1+f(2)}\right| = \left|T^{-1}\big(f(2)\big)\right| = \left|g\left(-\frac{1}{3}\right)\right| \le \frac{1}{3}.$$

197. If $f \equiv 0$, then clearly $f\overline{g}$ is holomorphic. If $f \not\equiv 0$, then there is $z_0 \in U$ such that $f(z_0) \ne 0$, continuity of $|f|$ implies there is $\delta > 0$ such that $f(z) \ne 0$, for all $z \in B = B(z_0, \delta) \subset U$, so $|g|^2 = \overline{g}g = \frac{(f\overline{g})g}{f}$ is also holomorphic on B. However, the only real-valued holomorhpism is constant function. So $|g|^2$ is constant. But $|g|$ is continuous, $|g|$ must be constant, so from Cauchy-Riemann equation, g is constant on B. In other words, the following equation holds

$$g\left(z_0 + \frac{1}{n}\right) = \text{const.}, \forall n \ge \left[\frac{1}{\delta}\right] + 1,$$

but $\frac{1}{n} \to 0$, so $g \equiv \text{const.}$ on U.

198. To be added.

199. To be added.

200. If f is constant function, let $f \equiv a$, then $a = 0$ or $a = 1$.

Suppose now f is nonconstant, then $f(0) = 0$ or $f(0) = 1$. Since $f(z) = \frac{2f(2z)}{1+f(2z)}$, taking $z = \frac{1}{4}, \frac{1}{8}, \ldots$, we can guess $f\left(\frac{1}{2^n}\right) = \frac{2^n}{2^n+1} = \frac{1}{1+\frac{1}{2^n}}$, this can proved by induction. Since $\frac{1}{2^n} \to 0$ and both f and $\frac{1}{1+z}$ are holomorphic on $B(0,1)$, $f(z) = \frac{1}{1+z}$ on $B(0,1)$.

To sum up, $f(z) = \frac{1}{1+z}$ is the only solution.

201. Let's write z in the problem by z'. Define $T = \frac{z+w}{1+\overline{w}z}, S = \frac{z+z'}{1+\overline{z'}z}$, then $g = S^{-1} \circ f \circ T : B(0,1) \to B(0,1)$ is holomorphic with $g(0) = S^{-1} \circ f \circ T(0) = S^{-1} \circ f(w) = S^{-1}(z') = 0$. So by Schwarz lemma, $|g'(0)| \le 1 \implies |(S^{-1})'(z')f'(w)T'(0)| \le 1$. Since

$$\begin{cases} S^{-1}(z) = \dfrac{z - z'}{1 - z\overline{z'}}, \\ T'(z) = \dfrac{1 + \overline{w}z - (z+w)\overline{w}}{(1+\overline{w}z)^2}, \end{cases} \implies \begin{cases} (S^{-1})'(z') = \dfrac{1}{1 - |z'|^2}, \\ T'(0) = 1 - |w|^2. \end{cases}$$

Plugging in all results, we have $|f'(w)| \le \dfrac{1-|z'|^2}{1-|w|^2}$.

202. To be added.

203. On the contrary let's suppose there is $z_0 \in U$ such that $|f(z_0)| > M$.

Let $\{\delta_n\}$ be a increasing sequence defined by $\delta_n = (1 - \frac{1}{n})d(z_0, \partial U)$, $n \geq 2$. Define $C_n = \partial B(z_0, \delta_n)$ and let $z_n \in C_n$ satisfies $f(z_n) = \max_{z \in \overline{B(z_0,\delta_n)}} |f(z)|$, then $\{|f(z_n)|\}$ is strictly increasing, hence $|f(z_n)| > |f(z_0)|$, for all $n \geq 2$, so $\overline{\lim_{n\to\infty}} |f(z_n)| \geq |f(z_0)| > M$, a contradiction.

Remark. *We extract from the second paragraph a fact that if there is z_0 in a domain U such that $|f(z_0)| > M$, then there is a sequence $\{z_n \in U\}$ converging to ∂U such that $\overline{\lim_{n\to\infty}} |f(z_n)| > M$ with the same construction as above.*

204. With the experience of problem 202 and the result in problem 203, this can be easily done. Consider $g(z) \triangleq f(z) \Big/ \prod_{k=1}^{n} \dfrac{z - z_k}{1 - \overline{z_k} z}$, then g is continuous on $B(0,1)$. Since f is holomorphic at z_i, so there are small enough δ_i's such that on $B_i = B(z_i, \delta_i) \subset B(0,1)$, $f|_{B_i} = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_i)}{n!}(z - z_i)^n$ and $B_i \cap B_j = \emptyset$ when $i \neq j$. It is clear that on each B_i, $\frac{f|_{B_i}}{(z-z_i)^{k_i}}$ is holomorphic, where k_i is the multiplicity of the root z_i, so g is holomophic on both $B(0,1) - \cup_i B_i$ and $\cup_i B_i$, hence g is holomorphic on $B(0,1)$.

Now since $|f| < 1$ and $\lim_{|z|\to 1} \prod_{k=1}^{n} |\frac{z - z_k}{1 - \overline{z_k} z}| = 1^n = 1$, so for any $\{z_n\}$ which converges to a point in the boundary of $B(0,1)$,

$$\overline{\lim_{n\to\infty}} |g(z_n)| \leq \frac{1}{1} = 1,$$

and hence $|g| \leq 1$.

Remark. *We extract from the first paragraph a useful fact, let U be a domain and $a_i \in U$, $i = 1, 2, \ldots, n$. If f is continuous on U, holomorphic on $U - \cup_{i=1}^{n}\{a_i\}$, then f is holomorphic on U.*

205. Let $z_0 \in \mathbb{C}$, then $f^{(n)}(z_0) = 0$, for some $n \geq 0$, hence $z_0 \in \cup_{n\geq 0} Z(f^{(n)})$. In other words, $\cup_{n\geq 0} Z(f^{(n)}) = \mathbb{C}$, but $\mathbb{C}$ is complete, it is of second category, hence there is $n \geq 0$ s.t. $(\overline{Z(f^{(n)})})^\circ = (Z(f^{(n)}))^\circ \neq \emptyset$, but then $f^{(n)} \equiv 0$, the result follows.

206. The only if ($\Rightarrow$) direction is clear. For the if direction, since f can only have countably many zeros in D, let $Z(f) = \{z_1, z_2, \ldots\}$ be the collection of its roots with order $2k_i$ repsectively. Then $g = \frac{f}{\prod_{i=1}(z - z_i)^{2k_i}}$ will be nonvanishing holomorphic function on D. As each root z_i is isolated, any z_i can be seen as a removable singularity of $g|_{D\setminus Z(f)}$ and hence g is holomorphic on each of the zero of f.

Claim. Any nonvanishing holomorphic function h on a simply connected domain D has a holomorphic sqaure root on D.

Proof. This is a simple observation that if f is nonvanishing on D, take $z_0 \in D$, then the function h defined by

$$h(z) = \int_{z_0}^{z} \frac{f'(w)}{f(w)}\, dw + \log f(z_0)$$

is a holomorphic branch of $\log f$ by fixing a value of $\log f(z_0)$. Now $F = e^{\frac{1}{2}h}$ is holomorphic and clearly $F^2 = f(z)$, hence F is desired square root. □

Finally by the last claim we see $g = \frac{f}{\prod_{i=1}(z - z_i)^{2k_i}} = G^2$ for some holomorphic function G on D, so $f = \left(G \prod_{i=1}(z - z_i)^{k_i}\right)^2$.

Remark. *The claim above can be modified to be "Any nonvanishing holomorphic function h on D has a holomorphic n-th root on D".*

207. (a) Consider $g = 1 + \sum_{n=2}^{\infty} n a_n z^n$, let $z \in \overline{B(0,1)}$, then since $|n a_n z^n| \le n|a_n|$, by the Weierstrass M-test $\sum_{n=1}^{N} n a_n z^n$ converges uniformly on $\overline{B(0,1)}$, and hence we can do termwise integration to get $f(z) = \int_0^z g(t)\,dt$. Which implies for each $z \in B(0,1)$, we have power series representation (with radius of convergence at least 1), so it is differentiable everywhere in D.

(b) Let $z_0 \in D$, then for each $r \in (|z_0|, 1)$, on $|z| = r$,

$$|f(z) - f(z_0) - (z - z_0)| = \left|\sum_{n\ge 2} a_n(z^n - z_0^n)\right| \le |z - z_0| \sum_{n \ge 2} n|a_n| r^n < |z - z_0|,$$

so $f(z) - f(z_0)$ and $z - z_0$ share the same number of root, which is 1.

208. $|\sum_{i=1}^n \epsilon_i z_i|^2 \le |\sum_{i=1}^n \epsilon_i w_i|^2$ implies

$$\sum_{i,j} \epsilon_i\epsilon_j z_i\overline{z_j} \le \sum_{i,j} \epsilon_i\epsilon_j w_i\overline{w_j}$$

$$\underbrace{\sum_{i=1}^n |z_i|^2}_{A} + \sum_{i<j} \epsilon_i\epsilon_j(z_i\overline{z_j} + \overline{z_i\overline{z_j}}) \le \underbrace{\sum_{i=1}^n |w_i|^2}_{B} + \sum_{i<j} \epsilon_i\epsilon_j(w_i\overline{w_j} + \overline{w_i\overline{w_j}})$$

$$A + \sum_{i=1}^{n-1} \epsilon_i \underbrace{\sum_{j=i+1}^n \epsilon_j 2\operatorname{Re}(z_i\overline{z_j})}_{A_i(\epsilon_{i+1},\dots,\epsilon_n)} \le B + \sum_{i=1}^{n-1} \epsilon_i \underbrace{\sum_{j=i+1}^n \epsilon_j 2\operatorname{Re}(w_i\overline{w_j})}_{B_i(\epsilon_{i+1},\dots,\epsilon_n)}$$

$$A + \sum_{i=1}^{n-1} \epsilon_i A_i \le B + \sum_{i=1}^{n-1} \epsilon_i B_i.$$

It is explicit that A_i and B_i are independent of $\epsilon_1, \dots, \epsilon_i$. Now if we take $(\epsilon_1, \epsilon_2, \dots, \epsilon_n) = (1, \epsilon_2, \dots, \epsilon_n)$ and $(\epsilon_1, \epsilon_2, \dots, \epsilon_n) = (-1, \epsilon_2, \dots, \epsilon_n)$, then one has respectively

$$\begin{cases} A + A_1(\epsilon_2, \dots, \epsilon_n) + \sum_{i=2}^{n-1} \epsilon_i A_i \le B + B_1(\epsilon_2, \dots, \epsilon_n) + \sum_{i=2}^{n-1} \epsilon_i B_i, \\ A - A_1(\epsilon_2, \dots, \epsilon_n) + \sum_{i=2}^{n-1} \epsilon_i A_i \le B - B_1(\epsilon_2, \dots, \epsilon_n) + \sum_{i=2}^{n-1} \epsilon_i B_i, \end{cases}$$

they add up to $A + \sum_{i=2}^{n-1} \epsilon_i A_i \le B + \sum_{i=2}^{n-1} \epsilon_i B_i$. Now all A_i and B_i are independent of ϵ_2, we repeat the process to get $A + \sum_{i=3}^{n-1} \epsilon_i A_i \le B + \sum_{i=3}^{n-1} \epsilon_i B_i$. We finally arrive to $A \le B$, as desired.

209. Let $K \subseteq B(0,1)$ be compact. Let $r = \sup\{|z| : z \in B(0,1)\}$, by compactness of K, $r = |z_0|$, for some $z_0 \in K$. Define $C = \partial B(0, \frac{1+r}{2})$, then C is also compact. Moreover, $d := \inf\{|c - k| : c \in C, k \in K\} = |c - k| > 0$, for some $c \in C, k \in K$. So for each $z \in K$,

$$\begin{aligned} |f_m'(z) - f_n'(z)| &= \frac{1}{2\pi}\left|\int_C \frac{f_m(w) - f_n(w)}{(w - z)^2}\,dw\right| \\ &\le \frac{1}{2\pi}\int_C \frac{|f_m(w) - f_n(w)|}{|w - z|^2}\,|dw| \\ &\le \frac{1}{2\pi} \cdot \frac{\sup_{w\in C}|f_m(w) - f_n(w)|}{d^2} \cdot 2\pi \cdot \frac{1+r}{2} \\ &< \frac{\sup_{w\in C}|f_m(w) - f_n(w)|}{d^2}. \end{aligned}$$

Since $\{f_n\}$ converges uniformly on every compact subset of $B(0,1)$ (in particular, C), so $\{f_n'\}$ converges uniformly on K (recall that $\{f_n\}$ converges uniformly on X iff it is uniformly Cauchy on X).

2.8 Real Analysis

210. Observe that both $g(x) = m^*(E\cap[-x,0])$ and $f(x) = m^*(E\cap[0,x])$ are continuous functions for $x \geq 0$. Let's check it for f, let $x \geq y$, then

$$f(x) = m^*(E\cap([0,y]\cup[y,x])) \leq m^*(E\cap[0,y]) + m^*(E\cap[y,x]) \implies f(x) - f(y) \leq |x-y|.$$

And the checking of g being continuous is the same.

Finally since one of $I = E\cap[0,+\infty)$ and $J = E\cap(-\infty,0]$ must be positive, let's say $I > 0$, then by continuity for any $c \in [0, m^*(E))$, there is $a \in [0,+\infty)$ such that $f(a) = c < \infty$.

211. Let $I_n = [n\frac{\epsilon}{2}, (n+1)\frac{\epsilon}{2})$, then $E\cap I_n$'s are disjoint, the convergence of $\sum_{n=1}^\infty m(E\cap I_n)$ and $\sum_{n=1}^\infty m(E\cap I_{-n})$ imply we can take N_1, N_2 such that

$$m\left(E\cap\cup_{n\geq N_2+1}I_{-n}\right), m\left(E\cap\cup_{n\geq N_1+1}I_n\right) < \epsilon,$$

and since each $m(A\cap I_n) \leq \frac{\epsilon}{2} < \epsilon$, the decomposition

$$E\cap\cup_{n\geq N_2+1}I_{-n}, \quad E\cap I_{-N_2}, \quad E\cap I_{-N_2+1}, \quad \ldots, E\cap I_{N_1}, \quad E\cap\cup_{n\geq N_1+1}I_n$$

will do.

212. Assume E is measurable, there there is an open set O and a closed set F such that $F\subseteq E\subseteq O$ and $m^*(O\setminus E), m^*(E\setminus F) < \frac{\epsilon}{2}$, then since $(O\setminus E)\cup(E\setminus F) = (O\cup E)\cap(O\setminus F)\cap(E\cap F)^c \supseteq O\cap(O\setminus F)\cap F^c = O\setminus F$, hence

$$m^*(O\setminus F) \leq m^*((O\setminus E)\cup(E\setminus F)) < \epsilon.$$

Conversely, observe that $E\supseteq F$, $O\setminus E\subseteq O\setminus F$ and use the outer measure property in the remark.

213. Let $A = \{x\in[a,b] : f(x) = g(x)\}$, $A' = [a,b]\setminus A$ (then $[a,b] = A\sqcup A'$ and $m(A') = 0$) and $a'\in A'$. We claim that

for any neighborhood U of a' that is open in $[a,b]$, there is an $a\in A$ such that $a\in U$.

Suppose it's wrong, that is, there is an open neighborhood U such that $A\cap U = \emptyset$, it follows that $x\in U \implies x\in A'$, this implies $U\subseteq A'$. However U is open in $[a,b]$, there is an open set U' in $\mathbb{R}$ such that $U = U'\cap[a,b]$, so $A'\supseteq U = U'\cap[a,b]\supseteq U'\cap(a,b)$, but $U'\cap(a,b)$ is open in $\mathbb{R}$ which doesn't have measure 0, contradiction.

Now by the claim we know that for each $a'\in A'$, there is a sequence $\{a_n\}$ in A such that $\lim_{n\to\infty} a_n = a'$, so

$$f(a') = f\left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} g(a_n) = g\left(\lim_{n\to\infty} a_n\right) = g(a'),$$

hence $f = g$ on $[a,b]$.

With the same proof $[a,b]$ can be replaced by (a,b), however, general measurable set cannot replace the interval. Let $x_0, x_1\in[a,b]$ such that $f(x_0) = g(x_0)$ but $f(x_1)\neq g(x_1)$. On $G = \{x_0\}\cup\{x_1\}$, $f = g$ almost everywhere (except x_1) and $f|_G$ and $g|_G$ are continuous (with respect to the subspace topology), but it doesn't imply $f|_G(x_1) = g|_G(x_1)$.

214. (a) Let's denote $A = \{x\in\mathbb{R} : x \text{ lies in infinitely many of } A_k\text{'s}\}$. Since x lying in infinitely many of E_k's is equivalent to say that for any $k\in\mathbb{N}$, there is a natural number $n\geq k$, $x\in E_n$. Hence

$$A = \bigcap_{k=1}^\infty\{x : \exists n\geq k, x\in E_n\} = \bigcap_{k=1}^\infty\bigcup_{n=k}^\infty\{x : x\in E_n\}.$$

(b) Since $A \subseteq \cup_{n=k}^{\infty} E_n$, for all $k \in \mathbb{N}$, hence

$$m(A) \leq m\left(\cup_{n=k}^{\infty} E_n\right) \leq \sum_{n=k}^{\infty} m(E_n),$$

by the convergence of $\sum m(E_n)$ in the hypothesis, we get $m(A) = 0$ by letting $k \to \infty$.

215. Give an $\epsilon > 0$, we note that

$$\begin{aligned} x \in [a,b] &\implies x \in \bigcap_{\epsilon>0} \bigcup_{N=1}^{\infty} \bigcap_{n\geq N} \{x \in [a,b] : |f(x) - f_n(x)| < \epsilon\} \\ &\implies x \in \bigcup_{N=1}^{\infty} \underbrace{\{x \in [a,b] : |f(x) - f_N(x)| < \epsilon\}}_{=E_N}, \end{aligned}$$

the logic above means exactly the inclusion $[a,b] \subseteq \cup_{N=1}^{\infty} E_N$. We note that $f - f_N$ is a continuous function, hence $E_N = \{x \in [a,b] : (f - f_N)(x) < \epsilon\} = \big((f-f_N)^{-1}\big((-\epsilon,\epsilon)\big)\big) \cap [a,b]$ is open in $[a,b]$. Since $[a,b]$ is compact, there is an integer N such that $[a,b] \subseteq \cup_{k=1}^{N} E_k$. Hence for all $x \in [a,b]$ (there is $k_x \in \{1,2,\ldots,N\}$, $x \in E_{k_x}$), when $n > N$,

$$|f(x) - f_n(x)| = f(x) - f_n(x) < f(x) - f_{k_x}(x) < \epsilon,$$

that is, the choice of ϵ is independent of x.

216. No! Let $A \subset [0,1)$ be nonmeasurable. Recall that any open interval is homeomorphic to the real line $\mathbb{R}$ (allowing one end of the interval to be be infinity), that is, there is a homeomorphism $h : \mathbb{R} \to (-1,0)$, we then construct $f : \mathbb{R} \to (-1,1)$ by

$$f(x) = \begin{cases} x, & x \in A, \\ h(x), & x \in \mathbb{R} \setminus A. \end{cases}$$

Clearly f is injective on both A and $\mathbb{R} \setminus A$, hence for each $c \in \mathbb{R}$, $f^{-1}(\{c\})$ is measurable, but $f^{-1}\big([0,+\infty)\big) = A$ is not measurable, meaning that f is not measurable.

217. Let $\epsilon > 0$ be given. Since E is of finite measure, by the first principle there is a finite disjoint union of bounded open intervals $U = \sqcup_{i=1}^{n} I_i$ such that $m(U\Delta E) < \epsilon$. Let's define $h = \chi_U = \sum_{i=1}^{n} \chi_{I_i}$ we claim that this h will do.

We try to prove that $h = \chi_E$ except a set of measure at most ϵ. Since $y \in \{x \in I : h(x) \neq \chi_E(x)\} \iff y \in I$ and $h(y) \neq \chi_E(y)$. However, $h(y) \neq \chi_E(y) \iff y \in (U \setminus E) \sqcup (E \setminus U)$, thus

$$\{x \in I : h(x) \neq \chi_E(x)\} = I \cap (U\Delta E).$$

Hence if we let $F = \{x \in I : h(x) = \chi_E(x)\}$, then $h = \chi_E$ on F and $m(I \setminus F) = m(I \cap (U\Delta E)) \leq m(U\Delta E) < \epsilon$.

218. Let $\psi = \sum_{i=1}^{n} a_i \chi_{A_i}$, $\sqcup_{i=1}^{n} A_i = I$, then for each χ_{A_i}, there is an $F_i \subseteq I$ and a step function h_i such that

$$h_i = \chi_{A_i} \text{ on } F_i \quad \text{and} \quad m(I \setminus F_i) < \frac{\epsilon}{n}.$$

Now we take $F = \cap_{i=1}^{n} F_i$, then $m(I \setminus F) < \epsilon$ and $h_i = \chi_{A_i}$ on F for $i = 1,2,\ldots,n$, so $\sum_{i=1}^{n} a_i h_i = \sum_{i=1}^{n} a_i \chi A_i$ on F. Finally, a linear combination of step function is still a step function.

219. As f is bounded measurable function, there is a simple function ψ such that $|f - \psi| < \epsilon$ on I. Define $\psi = \sum_{i=1}^{n} a_i \chi_{A_i}$, where $A_i = \psi^{-1}(a_i)$ and $a_i \neq 0$ for all i (if $a_i = 0$, the term $0 \cdot \chi_{\psi^{-1}(0)}$ is redundant). We now try to approximate ψ by a step function h. This is made easy with the help of problem 218, we can construct h such that $\psi = h$ on F and $m(I \setminus F) < \epsilon$, hence we are done.

220. As f is measurable, there is a sequence of simple functions $\{\phi_n\}$ such that $\phi_n \to f$ pointwise on E. By Egoroff's theorem, given an $\epsilon > 0$, there is a measurable subset (can be chosen to be closed) F of E such that $\phi_n \rightrightarrows f$ on F and $m(E \setminus F) < \epsilon$. Hence we are able to find an N such that $|f - \phi_N| \leq 1$ on F, this implies $|f| \leq 1 + |\phi_N|$ on F. But simple functions are always bounded, hence f is bounded on F with $m(E \setminus F) < \epsilon$.

221. Assume $f \overset{m}{\to} g$. $A \triangleq \{x \in E : f(x) \neq g(x)\} = \cup_{k=1}^{\infty}\{x \in E : |f(x) - g(x)| > \frac{1}{k}\}$. Observe that

$$|f(x) - g(x)| > \frac{1}{k} \implies \forall n \in \mathbb{N}, \frac{|f(x) - f_n(x)| + |f_n(x) - g(x)|}{2} > \frac{1}{2k},$$

this then implies for all $n \in \mathbb{N}$, $|f(x) - f_n(x)| > \frac{1}{2k}$ or $|f_n(x) - g(x)| > \frac{1}{2k}$, hence

$$A \subseteq \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \left(\left\{ x \in E : |f(x) - f_n(x)| > \frac{1}{2k} \right\} \cup \left\{ x \in E : |g(x) - g_n(x)| > \frac{1}{2k} \right\} \right).$$

Given $\frac{1}{2k}, \epsilon > 0$, there is an n_k such that both

$$m\left\{ x \in E : |f(x) - f_{n_k}(x)| > \frac{1}{2k} \right\}, m\left\{ x \in E : |g(x) - g_{n_k}(x)| > \frac{1}{2k} \right\} < \frac{\epsilon}{2^{k+1}},$$

observe also that $A \subseteq \bigcup_{k=1}^{\infty} \left(\left\{ x \in E : |f(x) - f_{n_k}(x)| > \frac{1}{2k} \right\} \cup \left\{ x \in E : |g(x) - g_{n_k}(x)| > \frac{1}{2k} \right\} \right)$, hence

$$m(A) \leq \sum_{k=1}^{\infty} \left(\frac{\epsilon}{2^{k+1}} + \frac{\epsilon}{2^{k+1}} \right) = \epsilon.$$

But $\epsilon > 0$ is arbitrary, $m(A) = 0$.

Conversely assume $f = g$ a.e. on E. We define $E_0 = \{x \in E : f(x) \neq g(x)\}$, then $m(E_0) = 0$ and

$$m\{x \in E : |g(x) - f_n(x)| > \eta\} \leq m\{x \in E : |f(x) - f_n(x)| > \eta\},$$

this implies $f_n \overset{m}{\to} g$.

222. The first one follows easily from the fact that $\big||f_n(x)| - |f(x)|\big| > \eta \implies |f_n(x) - f(x)| > \eta$.

The second one follows form the observation that $\frac{A+B}{2} > C \implies A > C$ or $B > C$.

The last one is a kind of complicated (maybe my proof is messy). Anyhow, it works! Observe that $f_n g_n - fg = f(g_n - g) + (f_n - f)(g_n - g) + g(f_n - f)$. Let $\eta, \epsilon > 0$ be given, then (everything is evaluated at x)

$$|f_n g_n - fg| > \eta \implies |f||g_n - g| + |f_n - f||g_n - g| + |g||f_n - f| > \eta,$$

this implies

$$|f||g_n - g| > \eta/3, \quad |g||f_n - f| > \eta/3 \quad \text{or} \quad \left(\begin{array}{c} |f_n - f||g_n - g| > \eta/3 \\ \Downarrow \\ |f_n - f| > \sqrt{\eta/3} \text{ or } |g_n - g| > \sqrt{\eta/3} \end{array} \right).$$

Let $P(x)$ be a property satisfied by x and we denote $m\{x \in E : P(x)\}$ by $m\{P\}$. For example, $m\{x \in E : f(x) > 0\} = m\{f > 0\}$. Then from the above logic,

$$m\{|f_n g_n - fg| > \eta\} \leq m\overbrace{\{|f||g_n - g| > \eta/3\}}^{\triangleq A} + m\overbrace{\{|g||f_n - f| > \eta/3\}}^{\triangleq B} \\ + m\{|f_n - f| > \sqrt{\eta/3}\} + m\{|g_n - g| > \sqrt{\eta/3}\}.$$

From hypothesis the last two terms can be small when n large. We apply problem 220 to get uniform bound of f and g on "large" enough subsets of E such that we can also estimate the first two terms. By problem 220, given $\epsilon/6 > 0$, we can find subsets $F, G \subseteq E$ such that there is $M > 0$,

$$|f| < M \text{ on } F,\ m(E \setminus F) < \epsilon/6 \quad \text{and} \quad |g| < M \text{ on } G,\ m(E \setminus G) < \epsilon/6.$$

That is to say, on F, $|f(x)||g_n(x) - g(x)| > \eta/3 \implies |g_n(x) - g(x)| > \eta/(3M)$. On G, $|g(x)||f_n(x) - f(x)| > \eta/3 \implies |f_n(x) - f(x)| > \eta/(3M)$, hence

$$\begin{aligned} m(A) + m(B) &= m(A \cap F) + m(A \setminus F) + m(B \cap G) + m(B \setminus G) \\ &< m\{x \in F : |g_n(x) - g(x)| > \eta/(3M)\} + m\{x \in G : |f_n(x) - f(x)| > \eta/(3M)\} + \epsilon/6 + \epsilon/6 \\ &\le m\{|g_n - g| > \eta/(3M)\} + m\{|f_n - f| > \eta/(3M)\} + 2 \cdot \epsilon/6, \end{aligned}$$

recall once the abbreviation is adopted, we mean all elements x in E having property $P(x)$. Now we can find an N such that when $n > N$, all 4 remaining terms become less than $\epsilon/6$, hence

$$m\{|f_n g_n - fg| > \eta\} < 6 \cdot \epsilon/6 = \epsilon.$$

223. ($\Rightarrow$) Assume $f_n \overset{m}{\to} f$, let $\{n_k : k \ge 1\} \subseteq \mathbb{N}$, then $f_{n_k} \overset{m}{\to} f$, hence having a further subsequence $\{f_{n_{k_p}}\}$ that converges pointwise to f a.e. on E.

($\Leftarrow$) Assume $f_n \overset{m}{\not\to} f$, then there are $\eta, \epsilon_0 > 0$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $m\{|f_{n_k} - f| > \eta\} \ge \epsilon_0$. But by hypothesis, there is a further subsequence $\{f_{n_{k_p}}\}$ such that $f_{n_{k_p}} \to f$ pointwise a.e. on E. By Egoroff's theorem we can find a measurable subset F of E such that $f_{n_{k_p}} \rightrightarrows f$ on F and $m(E \setminus F) < \epsilon_0/2$, now

$$\begin{aligned} \epsilon_0 \le m\{|f_{n_{k_p}} - f| > \eta\} &= m\left\{\{|f_{n_{k_p}} - f| > \eta\} \cap F\right\} + m\left\{\{|f_{n_{k_p}} - f| > \eta\} \setminus F\right\} \\ &< m\left\{\{|f_{n_{k_p}} - f| > \eta\} \cap F\right\} + \epsilon_0/2. \end{aligned}$$

However, by uniform convergence on F we can find a P such that $|f_{n_{k_P}} - f| \le \eta$ on F, and at $p = P$ the above inequality implies $\epsilon_0 < \epsilon_0/2$, a contradiction.

224. ($\Rightarrow$) If $f_n \overset{m}{\to} f$, then given $\epsilon > 0$, there is an N such that $n \ge N \implies m\{|f_n - f| > \frac{\epsilon}{2(m(E)+1)}\} < \epsilon/2$, hence when $n \ge N$,

$$\begin{aligned} \int_E \frac{|f_n - f|}{1 + |f_n - f|} &= \int_{\{|f_n - f| > \frac{\epsilon}{2(m(E)+1)}\}} \frac{|f_n - f|}{1 + |f_n - f|} + \int_{\{|f_n - f| \le \frac{\epsilon}{2(m(E)+1)}\}} \frac{|f_n - f|}{1 + |f_n - f|} \\ &\le m\left\{|f_n - f| > \frac{\epsilon}{2(m(E)+1)}\right\} + \frac{\frac{\epsilon}{2(m(E)+1)}}{1 + \frac{\epsilon}{2(m(E)+1)}} m(E) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2(m(E)+1)} m(E) \\ &< \epsilon. \end{aligned}$$

($\Leftarrow$) Assume $\lim_{n\to\infty} \rho(f_n, f) = 0$, then if $f_n \overset{m}{\not\to} f$, there are $\eta, \epsilon_0 > 0$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $m\{|f_{n_k} - f| > \eta\} \ge \epsilon_0$, hence

$$\int_E \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \ge \int_{\{|f_n - f| > \eta\}} \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} > \frac{\eta}{1 + \eta} \epsilon_0,$$

that's a contradiction.

225. Define for $n = 1, 2, \ldots$, $X_n = \left\{x \in E : \frac{\|f\|_\infty}{2^n} < |f(x)| \le \frac{\|f\|_\infty}{2^{n-1}}\right\}$, then $X_n \subseteq \left\{x \in E : \frac{\|f\|_\infty}{2^n} < |f(x)|\right\}$, hence

$$\int_{X_n} |f(x)| \le \frac{\|f\|_\infty}{2^{n-1}} m(X_n) \le \frac{\|f\|_\infty}{2^{n-1}} \cdot \frac{C}{\left(\frac{\|f\|_\infty}{2^n}\right)^\alpha} = 2C\|f\|^{1-\alpha} \frac{1}{(2^{1-\alpha})^n}.$$

Here X_n's are disjoint and $\sqcup_n X_n = E$.

226. By part (b) of problem 214, for almost all $x_0 \in E$, there is an index $K(x_0) \in \mathbb{N}$ such that $x \notin \{x \in E : |f_n(x)| > \alpha_n\}$ when $n > K(x_0)$. That is, whenever $n > K(x_0)$,

$$|f_n(x_0)| \le \alpha_n \implies -1 \le \frac{f_n(x_0)}{\alpha_n} \le 1.$$

The result follows from taking $\overline{\lim}$ and $\underline{\lim}$ in the inequality. And it is obvious from definition that $\underline{\lim} \le \overline{\lim}$.

227. Since $x \in [a, b] \implies x \in \{f > 0\} = \cup_{p \ge 1}\{f > \frac{1}{p}\}$, hence $[a, b] = \cup_{p\ge1} \overbrace{\{x : [a, b] : f(x) > \tfrac{1}{p}\}}^{\triangleq K_p}$. Let $K_1' = K_1$, $K_p' = K_p \setminus \cup_{i=1}^{p-1} K_i$, then $\cup_p K_p' = [a, b]$ and for all $p \ge 1$,

$$\int_{[a,b]} f\chi_{E_n} \ge \int_{K_p'} f\chi_{E_n} \ge \frac{1}{p}\int_{K_p'} \chi_{E_n} = \frac{1}{p} m(E_n \cap K_p),$$

this implies $\lim_{n\to\infty} m(E_n \cap K_p') = 0$ for each p. Now observe that

$$m(E_n) = m\big(E_n \cap (\cup_{p\ge1} K_p')\big) = \sum_{p=1}^{\infty} m(E_n \cap K_p') \le \sum_{p=1}^{P} m(E_n \cap K_p') + \sum_{p=P+1}^{\infty} m(K_p').$$

Let $\epsilon > 0$ be given, we can fix a P such that $\sum_{p=P+1}^\infty m(K_p') < \frac{\epsilon}{2}$, after that we can choose an N such that for each $p = 1, 2, \ldots, P$, when $n > N$, $m(E_n \cap K_p') < \frac{\epsilon}{2P}$, hence when $n > N$,

$$m(E_n) < P \cdot \frac{\epsilon}{2P} + \frac{\epsilon}{2} = \epsilon.$$

228. To apply Fatou's lemma $\int_E \underline{\lim}\, u_n \le \underline{\lim} \int_E u_n$, we need the integrand u_n to be nonnegative and measurable. So it is natural to consider $u_n = g - f_n$, hence

$$\int_E (g - f) = \int_E \underline{\lim}(g_n - f_n) \le \underline{\lim} \int_E (g_n - f_n) = \lim \int_E g_n - \overline{\lim} \int_E f_n,$$

the last equality follows from the fact that $\underline{\lim}(-a_n) = -\overline{\lim}\, a_n$. Cancelling $\int_E g$ on both sides, we have

$$\overline{\lim} \int_E f_n \le \int_E f. \qquad (*)$$

By letting $u_n = g + f_n$ this time, $\int_E g + \int_E f \le \int_E g + \underline{\lim} \int_E f_n$, combined with (*), we obtain

$$\overline{\lim} \int_E f_n \le \int_E f \le \underline{\lim} \int_E f_n \implies \lim \int_E f_n = \int_E f.$$

229. (a) Without loss of generality let's assume $f(x)$ is bounded on each of $x \in \mathbb{R}$. We also assume $f \ge 0$, then by integrability, there is a bounded measurable function $f_{\epsilon/2}$ with $f_{\epsilon/2} \le f$ and with finite support E_0 (i.e. $f|_{E_0} \ne 0$ and $f|_{\mathbb{R}\setminus E_0} \equiv 0$) such that

$$\int_{\mathbb{R}} |f - f_{\epsilon/2}| = \int_{\mathbb{R}} f - \int_{\mathbb{R}} f_{\epsilon/2} < \frac{\epsilon}{2}. \qquad (*)$$

On E_0 since $f_{\epsilon/2}$ is bounded and measurable, let's assume $m(E_0) > 0$, there is a simple function η such that $0 \le f_{\epsilon/2} - \eta < \frac{\epsilon}{2m(E_0)}$, we can also require $\eta|_{\mathbb{R}\setminus E_0} \equiv 0$ (hence its support is contained in E_0), and hence

$$\int_{\mathbb{R}} |f_{\epsilon/2} - \eta| = \int_{E_0} |f_{\epsilon/2} - \eta| < \frac{\epsilon}{2}. \tag{**}$$

Finally by (*) and (**),

$$\int_{\mathbb{R}} |f - \eta| \le \int_{\mathbb{R}} |f - f_{\epsilon/2}| + \int_{\mathbb{R}} |f_{\epsilon/2} - \eta| < \epsilon.$$

The same argument also shows that the last inequality holds when $m(E_0) = 0$ (in fact the second integral vanishes). And the extension to general measurable function is obvious.

(b) We also do that for f nonnegative, and its extension to general measurable function is obvious. By part (a) there is a simple function η with finite support E_0 such that

$$\int_{\mathbb{R}} |f - \eta| < \frac{\epsilon}{2}.$$

Let's assume I is a closed interval such that $I \supseteq E_0$, then on I since η is simple. Let also $|\eta| < M$ for some $M > 0$, by problem 218, given any $\epsilon > 0$, we can find a step function s on I and measurable subset F of I such that

$$\eta = s \text{ on } F \quad \text{and} \quad m(I \setminus F) < \frac{\epsilon}{4M}.$$

Recall that such simple functions can be chosen such that the maximum of s is the maximum of η, let's assume also $|s| < M$, define $\hat{s} = s\chi_I$, then $\hat{s}$ is also a step function such that

$$\begin{aligned}\int_{\mathbb{R}} |f - \hat{s}| &\le \int_{\mathbb{R}} |f - \eta| + \int_{\mathbb{R}} |\eta - \hat{s}| < \frac{\epsilon}{2} + \int_{I\setminus F} |\eta - s| + \int_F |\eta - s| \\ &< \frac{\epsilon}{2} + 2Mm(I \setminus F) < \epsilon.\end{aligned}$$

(c) Just linearize the step function in part (b).

230. To be added.

231. ($\Rightarrow$) Since E has measure zero, for each $k \in \mathbb{N}$, there must be an open set $\mathcal{O}_k$ such that $\mathcal{O}_k \supseteq E$ with $m(\mathcal{O}_k) < \frac{1}{2^k}$. Write $\mathcal{O}_k = \sqcup_{p=1}^{\infty} I_p^{(k)}$ (in case if $\mathcal{O}_k$ is a finite union of disjoint open sets, let $I_p^{(k)}$ with large enough p be empty set). Then clearly for each p, $m(I_p^{(k)}) < m(\mathcal{O}_k) < \frac{1}{2^k}$. We claim that the collection $\mathcal{I} = \{I_p^{(k)} : p, k \ge 1\}$ will do.

Let $e \in E$, then $e \in I_{p_1}^{(1)}$, for some p_1. We can find k_2 such that $\frac{1}{2^{k_2}} < m(I_{p_1}^{(1)})$, and there is p_2 such that $e \in I_{p_2}^{(k_2)}$. Since $m(I_{p_2}^{(k_2)}) < \frac{1}{2^{k_2}} < m(I_{p_1}^{(1)})$, they must be distinct intervals. Inductively we can take k_{n+1} such that $\frac{1}{2^{k_{n+1}}} < m(I_{p_n}^{(k_n)})$ and find a p_{n+1} such that $e \in I_{p_{n+1}}^{(k_{n+1})}$, hence $e \in I_{p_1}^{(1)}, I_{p_n}^{(k_n)}$ for $n \ge 2$ and they are all different intervals.

Finally we check that $\sum_{I\in\mathcal{I}} m(I) \le \sum_{k=1}^{\infty} m(\mathcal{O}_k) = 1 < \infty$.

($\Leftarrow$) Since $x \in E$ implies $x \in \cap_{k\ge1} \cup_{n\ge k} I_n$, i.e. $x \in \cup_{n\ge k} I_n$, for all k, hence $m(E) \le m(\cup_{n\ge k} I_n) \le \sum_{n=k}^{\infty} m(I_n) \to 0$.

232. Let $e \in E$, $\dfrac{f(e+t)-f(e)}{t} = \sum_{k=1}^{\infty} \dfrac{\lambda\big((c_k,d_k)\cap[e,e+t)\big)}{t}$. Let $e \in (c_{k_p}, d_{k_p})$ for $p = 1,2,\ldots$, for each $n \geq 1$,

$$\frac{f(e+t)-f(e)}{t} \geq \sum_{p=1}^{n} \frac{\lambda\big((c_{k_p},d_{k_p})\cap[e,e+t)\big)}{t}.$$

We can take $t_n \in (0, \frac{1}{n})$ small such that $e + t_n < d_{k_p}$, for $p = 1,2,\ldots,n$, it follows that

$$\frac{f(e+t_n)-f(e)}{t_n} \geq \frac{n\cdot t_n}{t_n} = n,$$

but this is true for all n, hence $\lim_{n\to\infty} \dfrac{f(e+t_n)-f(e)}{t_n} = \infty$.

233. (a) Since

$$\left|\frac{f(x+h)-f(x)}{h}\right| \leq \frac{TV(f_{[x,x+h]})}{h} = \frac{TV(f_{[a,x+h]})-TV(f_{[a,x]})}{h}$$
$$= \frac{v(x+h)-v(x)}{h},$$

this implies $|f'| \leq v'$, hence $\int_a^b |f'| \leq \int_a^b v' \leq v(b)-v(a) = TV(f)$. The last inequality follows from the fact that v is increasing.

(b) Assume the equality holds, then since $\int_u^v |f'| \leq TV(f_{[u,v]})$ and

$$\left(\int_a^x |f'| - TV(f_{[a,x]})\right) + \left(\int_x^b |f'| - TV(f_{[x,b]})\right) = 0,$$

but both of them are nonpositive, hence $\int_a^x |f'| = TV(f_{[a,x]})$ for all $x \in [a,b]$, it follows that $\int_u^v |f'| = \int_a^v |f'| - \int_a^u |f'| = TV(f_{[u,v]})$.

For each $\epsilon > 0$ there is a $\delta > 0$ such that whenever $m(E) < \delta$, $\int_E |f'| < \epsilon$. Hence if $\{(c_i,d_i)\}_{i=1}^n$ is a disjoint collection of open intervals in $[a,b]$ with $\sum_{i=1}^n (d_i - c_i) < \delta$, then

$$\sum_{i=1}^n |f(d_i)-f(c_i)| \leq \sum_{i=1}^n TV(f_{[c_i,d_i]}) = \sum_{i=1}^n \int_{c_i}^{d_i} |f'| = \int_{\sqcup_{i=1}^n (c_i,d_i)} |f'| < \epsilon.$$

Conversely if f is absolutely continuous, then $f(x) = f(a) + \int_a^x f'$. It is enough to show $TV(f) \leq \int_a^b |f'|$. This is obvious since for every partition P of $[a,b]$, $P = \{a_0, a_1, \ldots, a_n\}$ with $a = a_0 < a_1 < a_2 < \cdots < a_n = b$,

$$\sum_{i=1}^n |f(a_i)-f(a_{i-1})| = \sum_{i=1}^n \left|\int_{a_{i-1}}^{a_i} f'\right| \leq \sum_{i=1}^n \int_{a_{i-1}}^{a_i} |f'| = \int_a^b |f'|,$$

and the result follows from taking supremum over all possible partitions.

234. If $m^*(A) = \infty$, we are done. Suppose $m^*(A) < \infty$, then given $\epsilon > 0$ we can find an open $U \supseteq A$ such that $m(U) < m^*(A) + \epsilon$. A simple if-then statement shows the following inclusion

$$f(B(a,r)) \subseteq B(f(a), Lr),$$

for any $a \in \mathbb{R}$ and $r > 0$. Hence by writing $U = \sqcup(a_i,b_i)$, then $A = U \cap A = \sqcup((a_i,b_i)\cap A)$, we see that $f((a_i,b_i)\cap A) \subseteq f((a_i,b_i)) = f\big(B(\frac{a_i+b_i}{2}, \frac{b_i-a_i}{2})\big) \subseteq B\left(f(\frac{a_i+b_i}{2}), L(\frac{b_i-a_i}{2})\right)$, and hence

$$\begin{aligned} f(A) = f\big(\sqcup_i ((a_i, b_i) \cap A)\big) &\subseteq \cup_i B\left(f(\tfrac{a_i+b_i}{2}), L(\tfrac{b_i-a_i}{2})\right) \\ m^*\left(f(A)\right) &\le m^*\left(\cup_i B\left(f(\tfrac{a_i+b_i}{2}), L(\tfrac{b_i-a_i}{2})\right)\right) \\ &\le \textstyle\sum_i m^*\left(B\left(f(\tfrac{a_i+b_i}{2}), L(\tfrac{b_i-a_i}{2})\right)\right) \\ &= \textstyle\sum_i L(b_i - a_i) = Lm(U). \end{aligned}$$

It follows that

$$m^*\left(f(A)\right) \le Lm(U) \le Lm^*(A) + L\epsilon,$$

we let $\epsilon \to 0^+$ to complete the proof.

Assume A is bounded and measurable. It is obvious that f is continuous, hence for any compact K, $f(K)$ is compact and hence measurable. By measurability of A there is a F_σ set $F = \cup_i K_i \subseteq A$ such that each K_i is compact and $m(A \setminus F) = 0$. Observe that $A = F \cup (A \setminus F)$, this implies

$$f(A) = f(F) \cup f(A \setminus F),$$

by what we have just proved, $f(A \setminus F)$ is measurable since it has measure zero. $f(F) = \cup_i f(K_i)$ is a countable union of closed sets which is also measurable.

It follows that f maps measurable set E to measurable set $f(E) = \cup_{n\in\mathbb{N}} f(E \cap [-n, n])$, a countable union of measurable sets.

235. As f, g are integrable, there is $E_0 \subseteq E$ such that f, g are finite on $E \setminus E_0$ with $m(E_0) = 0$. On $E \setminus E_0$, since $f(x)g(x) \ge 1$, then $f(y)g(x) \ge \frac{f(y)}{f(x)}$, for any x, y on $E \setminus E_0$. And by interchanging x, y in the inequality, we get $f(x)g(y) \ge \frac{f(x)}{f(y)}$. Adding them up and applying AM-GM inequality once,

$$f(y)g(x) + f(x)g(y) \ge \frac{f(y)}{f(x)} + \frac{f(x)}{f(y)} \ge 2.$$

Now the result follows by integrating respect to x and y respectively.

236. Let $\epsilon > 0$ be given. Assume Lusin's theorem holds on a finite measure space. Let f be real-valued measurable on E with $m(E) = \infty$. Define $E_n = E \cap (n, n+1)$, where $n \in \mathbb{Z}$, then we can find a continuous $g_n : \mathbb{R} \to \mathbb{R}$ and a measurable (in fact closed in the theorem) $F_n \subseteq E_n$ such that

$$f = g_n \text{ on } F_n \quad \text{and} \quad m(E_n \setminus F_n) < \epsilon/2^{|n|+3}.$$

Clearly

$$m(E \setminus \sqcup_n F_n) = \sum_{i\in\mathbb{Z}} m(E_i \setminus \sqcup_n F_n) \le \sum_{i\in\mathbb{Z}} m(E_i \setminus F_i) < \epsilon\left(1/2^3 + 1/2^3 + 1/2^3\right) < \epsilon/2.$$

We construct $g = \sum_{i\in\mathbb{Z}} g_i \chi_{F_i}$, then $g|_{\sqcup_n F_n} = f|_{\sqcup_n f_n}$, we claim that $g : \sqcup_n F_n \to \mathbb{R}$ is continuous with respect to the subspace topology. Fix a $x \in \sqcup_n F_n$, then there is an n such that $x \in F_n$. Hence by continuity of g_n on $\mathbb{R}$, given $\varepsilon > 0$ there is a $\delta > 0$ such that $g_n(B(x, \delta)) \subseteq B(g_n(x), \varepsilon)$. Since $x \in F_n \subseteq (n, n+1)$, we can choose $\delta > 0$ small such that $B(x, \delta) \subseteq (n, n+1)$, therefore $B(x, \delta) \cap (\sqcup_k F_k) = B(x, \delta) \cap F_n$, hence

$$\begin{aligned} g(B(x, \delta) \cap \sqcup_k F_k) &= g(B(x, \delta) \cap F_n) \\ &= g_n(B(x, \delta) \cap F_n) \\ &\subseteq B(g_n(x), \varepsilon) \\ &= B(g(x), \varepsilon), \end{aligned}$$

this proves our claim. Now measurability of $\sqcup_n F_n$ enables us to construct its closed subset F such that $m(\sqcup_n F_n \setminus F) < \epsilon/2$, thus

$$m(E \setminus F) \le m(E \setminus \sqcup_n F_n) + m(\sqcup_n F_n \setminus F) < \epsilon.$$

It is clear now that $g|_F$ is continuous w.r.t. subspace topology of F, so we are done with the help of problem 25.

237. We only prove the general case. By the simple approximation theorem, there is a sequence of simple functions $\{\phi_n\}$ such that $\phi_n \to f$ pointwise on $\mathbb{R}$ and $|\phi_n(x)| \le |f(x)|$. The integral comparison test shows that each ϕ_n is integrable and by Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi_n(x) g(tx)\, dm = \int_{\mathbb{R}} f(x) g(tx)\, dm \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}} \phi_n(x)\, dm = \int_{\mathbb{R}} f(x)\, dm.$$

Hence it suffices to prove the given identity holds for simple function, to this end, it suffices to prove it holds for characteristic function which has finite support (as the above convergence is uniform, interchange of limit processes is permitted).

Let A be measurable and $m(A) < \infty$, we will prove that

$$\lim_{t\to\infty} \int_{\mathbb{R}} \chi_A(x) g(tx)\, dm = \frac{m(A)}{T} \int_{[0,T]} g(x)\, dm. \tag{*}$$

Since $m(A) < \infty$, there is a descending collection of open set $\{\mathcal{O}_n\}$ for which $\mathcal{O}_n \supseteq A$ and $m(\cap \mathcal{O}_n \setminus A) = 0$, hence

$$\lim_{t\to\infty} \int_{\mathbb{R}} \chi_A(x) g(tx)\, dm = \lim_{t\to\infty} \int_A g(tx)\, dm = \lim_{t\to\infty} \int_{\cap \mathcal{O}_n} g(tx)\, dm = \lim_{t\to\infty} \lim_{n\to\infty} \int_{\mathcal{O}_n} g(tx)\, dm,$$

since g is bounded, the convergence $\lim_{n\to\infty} \int_{\mathcal{O}_n} g(tx)\, dm$ is then uniform, which implies it is enough to prove the limit

$$\lim_{t\to\infty} \int_{\mathcal{O}_n} g(tx)\, dm(x)$$

exists and is equal to $\frac{m(\mathcal{O}_n)}{T} \int_0^T g(x)\, dm$ (as the limit process can be interchanged). This motivates us to prove (*) is true when $A = (a, b)$, an bounded open interval (we will transit the result to open set). Consider when $A = (a, b)$, we aim to show $\lim_{t\to\infty} \int_a^b g(tx)\, dm = \frac{b-a}{T} \int_0^T g(x)\, dm$. We first observe that

$$I(t) := \int_a^b g(tx)\, dm(x) = \int_{\mathbb{R}} g(tx) \chi_{(a,b)}(x)\, dm(x) = \frac{1}{t} \int_{\mathbb{R}} g(x) \chi_{(a,b)}(x/t)\, dm(x) = \frac{1}{t} \int_{ta}^{tb} g(x)\, dm(x).$$

We take t large so that $t(b-a) > 2T$, then we can choose $n_1(t), n_2(t) \in \mathbb{Z}$ so that

$$ta \le n_1(t)T < (n_1(t)+1)T < \cdots < n_2(t)T \le tb \tag{**}$$

with

$$n_1(t)T - ta < T \quad \text{and} \quad tb - n_2(t)T < T. \tag{***}$$

Now

$$\begin{aligned} I(t) &= \frac{1}{t} \sum_{k=n_1(t)}^{n_2(t)-1} \int_{kT}^{(k+1)T} g(x)\, dm + \underbrace{\frac{\left(\int_{ta}^{n_1(t)T} + \int_{n_2(t)T}^{tb}\right) g(x)\, dm}{t}}_{:=R(t)} \\ &= \frac{1}{t} \sum_{k=n_1(t)}^{n_2(t)-1} \int_0^T g(x)\, dm + R(t) \\ &= \frac{(n_2(t) - n_1(t))T}{t(b-a)} \cdot \frac{(b-a)}{T} \int_0^T g(x)\, dm + R(t). \end{aligned} \tag{****}$$

Since $R(t) \leq \dfrac{2\int_0^T |g(x)|\,dm}{t}$, $\lim_{t\to\infty} R(t) = 0$. Moreover, by (**) and (***),

$$1 \geq \frac{(n_1(t) - n_2(t))T}{t(b-a)} \geq \frac{t(b-a) - 2T}{t(b-a)} = 1 - \frac{2T}{t(b-a)},$$

these shows, continued from (****) that

$$\lim_{t\to\infty} \int_a^b g(tx)\,dm(x) = \lim_{t\to\infty} I(t) = \frac{(b-a)}{T} \int_0^T g(x)\,dm.$$

Thus for open set $\mathcal{O}$ with $m(\mathcal{O}) < \infty$, $\mathcal{O} = \sqcup_{i=1}^{\infty} I_i$ where I_i is bounded open interval (in case when $\mathcal{O}$ is just a finite union, we let the extra indexed intervals be $\emptyset$), once again by boundedness of g, the convergence $\lim_{k\to\infty} \sum_{i=1}^k \int_{I_i} g(tx)\,dm$ is uniform, thus

$$\begin{aligned}
\lim_{t\to\infty} \int_{\mathcal{O}} g(tx)\,dm &= \lim_{t\to\infty} \lim_{k\to\infty} \sum_{i=1}^k \int_{I_i} g(tx)\,dm = \lim_{k\to\infty} \lim_{t\to\infty} \sum_{i=1}^k \int_{I_i} g(tx)\,dm \\
&= \lim_{k\to\infty} \sum_{i=1}^k \frac{m(I_i)}{T} \int_0^T g(x)\,dm \\
&= \frac{m(\mathcal{O})}{T} \int_0^T g(x)\,dm,
\end{aligned}$$

it follows, continued from the previous result, that

$$\begin{aligned}
\lim_{t\to\infty} \int_{\mathbb{R}} \chi_A(x) g(tx)\,dm &= \lim_{t\to\infty} \lim_{n\to\infty} \int_{\mathcal{O}_n} g(tx)\,dm(x) \\
&= \lim_{n\to\infty} \lim_{t\to\infty} \int_{\mathcal{O}_n} g(tx)\,dm(x) \\
&= \lim_{n\to\infty} \frac{m(\mathcal{O}_n)}{T} \int_0^T g(x)\,dm \\
&= \frac{m(A)}{T} \int_0^T g(x)\,dm.
\end{aligned}$$

238. To be added.

2.9 Fourier Analysis

239. To be added

240. To be added

241. To be added

242. To be added

243. To be added

244. We first show existence. Pick $g_n \in A$ so that $\inf\{\|f - g\| : g \in A\} = \lim_{n\to\infty} \|f - g_n\|$. If $\{g_n : n \geq 1\}$ is finite, then we are done as one of the g_i's must satisfy the the equality in the above limit. Let's assume $\{g_n : n \geq 1\}$ is infinite. Starting from $i = 2$, if g_i can be expressed as a linear combination of $\{g_1, \ldots, g_{i-1}\}$, we weed it out, and the remaining list of elements $\{g_{n_1}, g_{n_2}, \ldots\}$ can be made into an orthonormal sequence of functions. Explicitly,

$$e_1 = \frac{g_{n_1}}{\|g_{n_1}\|}, \quad e_k = \frac{g_{n_k} - \sum_{j=1}^{k-1} \langle g_{n_j}, e_j \rangle e_j}{\|g_{n_k} - \sum_{j=1}^{k-1} \langle g_{n_j}, e_j \rangle e_j\|}.$$

Since $f \in L^2(Q)$, the sequence $\{\sum_{k=1}^n \hat{f}(k)e_k\}_{k=1}^\infty$ is Cauchy in $L^2(Q)$, so this sequence converges to a $F \in L^2(Q)$ by completeness.

We claim that $F := \sum_{k=1}^\infty \hat{f}(k)e_k$ will do. Now we still have $\lim_{k\to\infty} \|f - g_{n_k}\| = \inf\{\|f - g\| : g \in A\}$, and since $g_{n_N} = \sum_{i=1}^N c_{Ni}e_i$,

$$\|f - F\| \le \left\|f - \sum_{k=1}^N \hat{f}(k)e_k\right\| + \left\|\sum_{k>N} \hat{f}(k)e_k\right\| \le \left\|f - \sum_{k=1}^N c_{Nk}e_k\right\| + \sum_{k>N} |\hat{f}(k)|^2$$
$$= \|f - g_{n_N}\| + \sum_{k>N} |\hat{f}(k)|^2,$$

the result follows from taking $N \to \infty$, since $e_k \in A, \forall k \implies F \in A$, thus existence is established.

Finally we have to show that such F is unique, so that Pf is well-defined. Assume both $F, G \in A$ are closest to f, the polynomial $P(\epsilon) := \|f - \epsilon F - (1-\epsilon)G\|^2$ has minimum at $\epsilon = 0$ and $\epsilon = 1$, hence $P'(0) = P'(1) = 0$, which implies $P'(\epsilon) \equiv 0$. Since

$$P(\epsilon) = \|f - G + \epsilon(G - F)\|^2 = \|f - G\|^2 + \epsilon^2\|G - F\|^2 + 2\epsilon \operatorname{Re}\langle f - G, G - f\rangle,$$

$P''(\epsilon) = 0 \implies \|G - F\| = 0$, i.e., $F = G$ a.e. on Q.

245. Let $g \in A$, we use the same trick as before, $k_1(\epsilon) := \|f - P_Af + \epsilon g\|^2$ is a polynomial in ϵ for which minima occurs at $\epsilon = 0$, hence $k_1'(\epsilon) = 0$. Since $k_1(\epsilon) = \|f - P_Af\|^2 + \epsilon^2\|g\|^2 + 2\epsilon \operatorname{Re}\langle f - P_Af, g\rangle$, $k_1'(0) \implies \operatorname{Re}\langle f - P_Af, g\rangle = 0$.

Similarly, $k_2(\epsilon) := \|f - P_Af + i\epsilon g\|^2 = \|f - P_Af\|^2 + \epsilon^2\|g\|^2 + 2\epsilon \operatorname{Im}\langle f - P_Af, g\rangle$ is minimum at $\epsilon = 0$, $k_2'(0) = 0 \implies \operatorname{Im}\langle f - P_Af, g\rangle = 0$, we conclude $\langle f - P_Af, g\rangle = 0$.

246. For part (a), let $f \in L^2(Q)$. By definition, $P_Af - P_A^2 f \in A$, and since $P_Af - P_A(P_Af) \in A^\perp$, we have $P_A = P_A^2$; For part (b), $\langle f_1, P_Af_2\rangle = \langle P_Af_1 + (f_1 - P_Af_1), P_Af_2\rangle = \langle P_Af_1, P_Af_2\rangle$, and similarly,

$$\langle P_Af_1, f_2\rangle = \langle P_Af_1, P_Af_2 + f_2 - P_Af_2\rangle = \langle P_Af_1, P_Af_2\rangle;$$

For part (c), $\|f\|^2 = \|P_Af + (f - P_Af)\|^2 = \|P_Af\|^2 + \|f - P_Af\|^2 \ge \|P_Af\|^2$; For part (d), when $f \in A$, $f - P_Af \in A \cap A^\perp$, so $P_Af = f$. When $f \in A^\perp$, $P_Af = f - (f - P_Af) \in A \cap A^\perp$, so $P_Af = 0$.

It remains to show P_A is linear. For each $\alpha \in \mathbb{C}$ and $f, g \in L^2(Q)$,

$$A \ni P_A(f + \alpha g) - P_Af - \alpha P_Ag = [P_A(f + \alpha g) - f - \alpha g] + [f - P_Af] + \alpha[g - P_Ag] \in A^\perp.$$

247. To be added

248. Assume $T \in [L^2(Q)]^*$. Since T is continuous,

$$A := \ker T$$

is closed subspace of $L^2(Q)$. If $T \equiv 0$, then we are done. Assume $T \not\equiv 0$, then there must be a $h \in L^2(Q)$ so that $Th \neq 0$. In other words, $h \notin A$, and hence $u := h - P_Ah \in A^\perp \setminus \{0\}$.

We observe that $\dim A^\perp = 1$. For each $g \in A^\perp$,

$$T\left(g - \frac{Tg}{Tu}u\right) = Tg - \frac{Tg}{Tu}Tu = 0,$$

which means that $g - \frac{Tg}{Tu}u \in A \cap A^\perp$ and hence $g = \frac{Tg}{Tu}u$. So $\operatorname{span}_{\mathbb{C}}\{u\} = A^\perp$.

Finally for each $f \in L^2(Q)$, $f - P_A = \frac{T(f-P_Af)}{Tu}u \implies \langle f - P_Af, u\rangle = \frac{Tf}{Tu}\|u\|^2$, and hence

$$\begin{aligned} Tf &= \frac{Tu}{\|u\|^2}\langle f - P_Af, u\rangle = \frac{Tu}{\|u\|^2}(\langle f, u\rangle - \langle P_Af, u\rangle) \\ &= \frac{Tu}{\|u\|^2}(\langle f, u\rangle - \underbrace{\langle f, P_Au\rangle}_{=0 \text{ as } u \in A^\perp}) = \left\langle f, \overline{\left(\frac{Tu}{\|u\|^2}\right)u}\right\rangle, \end{aligned}$$

we conclude $Tf = \langle f, g\rangle$, for some $g \in L^2(Q)$. Conversely, any linear functional of this form is bounded.

249. To be added

250. To be added

251. To be added

252. To be added

253. To be added

254. To be added

2.10 Number Theory

255. Since $(2, n) = 1$, $2a_1, 2a_2, \ldots, 2a_{\phi(n)}$ also form a reduced residue system modulo n, hence $a_i \equiv 2a_{f(i)} \pmod n$ for $i = 1, 2, \ldots, \phi(n)$, here f is a bijection from $\{1, 2, \ldots, \phi(n)\}$ to $\{1, 2, \ldots, \phi(n)\}$. Finally, we make use of the formula $\sin 2x = 2 \sin x \cos x$ and let $2a_j = h_in + a_i$, we are done.

256. Observe that $(4a^2 - 1)^2 = (4a^2 - 1)(4a^2 - 1) = (4a^2 - 1)(2a + 1)(2a - 1)$. Now in modulo $4ab - 1$,

$$(4a^2-1)^2 \equiv (4a^2-1)(4ab+2b)(4ab-2b) \equiv (4a^2-1)(1+2b)(1-2b) \equiv (4a^2-1)(4b^2-1) \equiv 0, \quad (\text{mod } 4ab-1)$$

on expansion and making use of the fact that $4ab \equiv 1 \pmod{4ab - 1}$ again, we can deduce that the original divisibility actually implies $4ab - 1 | (a - b)^2$, this is equivalent to say that $\frac{(a - b)^2}{4ab - 1} \in \mathbb{N}$.

Define $S = \left\{(x, y) \in \mathbb{N}^2 : \frac{(x - y)^2}{4xy - 1} \in \mathbb{N}, x \neq y\right\}$, We now suppose, for the sake of contradiction, there is a solution $(x', y') \in S$ with $x' \neq y'$, then S is not empty, clearly there is a smallest a and b such that $(a, b) \in S$, let's say $\frac{(a - b)^2}{4ab - 1} = k \in \mathbb{N}$, then by rearranging terms in this equation into a quadratic equation of a, we see that if a is a solution, then from product of root, $a' = \frac{b^2 + k}{a} \in \mathbb{N}$ is another solution, by the minimality of a, we deduce that $a' \geq a$, this implies $\frac{(a - b)^2}{4ab - 1} = k \geq b^2 - a^2$, this implies $a - b \geq (a + b)(4ab - 1)$, a contradiction.

257. To be added.

258. To be added.

259. To be added.

260. To be added.

261. To be added.

262. To be added.

263. To be added.

264. To be added.

265. To be added.

266. To be added.

267. To be added.

268. To be added.

269. (a) Suppose that (u, v) is the smallest positive solution of $x^2 - Dy^2 = 1$, then we need to show that $u \geq x_2$ (this implies $v \geq y_2$ as $u^2 - x_2^2 = D(v^2 - y_2^2)$), then we write

$$x_2 + \sqrt{D}y_2 = (u + \sqrt{D}v)^k,$$

for some $k \in \mathbb{N}$.

If $k = 2n$, $n \geq 1$, then $x_1 + \sqrt{D}y_1 = (u + \sqrt{D}v)^n$, this is a contradiction as

$$(x_1 + \sqrt{D}y_1)(x_1 - \sqrt{D}y_1) = (u + \sqrt{D}v)^n(u - \sqrt{D}v)^n \implies -1 = 1.$$

So k must be odd and write $k = 2n + 1, n \geq 0$, then we have

$$(x_1 + \sqrt{D}y_1)^2 = (u + \sqrt{D}v)^{2n+1} \iff (x_1 + \sqrt{D}y_1)^2(u - \sqrt{D}v)^{2n} = (u + \sqrt{D}v),$$

this is the same as $(x_1 + \sqrt{D}y_1)(u - \sqrt{D}v)^n = \sqrt{u + \sqrt{D}v}$, so there must be some $a, b \in \mathbb{Z}$ such that

$$\sqrt{u + \sqrt{D}v} = a + \sqrt{D}b \iff \boxed{u + \sqrt{D}v = (a + \sqrt{D}b)^2}. \qquad (*)$$

Since $2ab = v > 0$, so a, b must be both positive or negative, but since $\sqrt{u + \sqrt{D}v} > 0$, a, b are both positive. Next

$$(a + \sqrt{D}b)^2(a - \sqrt{D}b)^2 = (u + \sqrt{D}v)(u - \sqrt{D}v) = 1$$
$$\implies (a + \sqrt{D}b)(a - \sqrt{D}b) = d, d = \pm 1.$$

Case 1. If $d = 1$, then since from (*), $u = a^2 + Db^2 > a$, so contradiction arises as (u, v) are smallest such solution. We must have $d = -1$.

Case 2. When $d = -1$, then by the minimality, $a \geq x_1$ and $b \geq y_1$, as a result,

$$u = a^2 + Db^2 \geq x_1^2 + Dy_1^2 = x_2.$$

(b) It is known that (x_2, y_2) is smallest integer solution of $x^2 - Dy^2 = 1$, so all solutions are given by (u_n, v_n) defined by

$$u_n + \sqrt{D}v_n = (x_2 + \sqrt{D}y_2)^n = (x_1 + \sqrt{D}y_1)^{2n},$$

and hence (u_n', v_n') defined by $u_n' + \sqrt{D}v_n' = (x_1 + \sqrt{D}y_1)^{2n+1}$ will give all solutions of $x^2 - Dy^2 = -1$.

270. To be added.

271. Clearly when $n = 1, a = b = c = 2$ is a solution.

We will show that the equation has no solution when $n \geq 2$, we first see that a, b can't be both even for such n. If they are even (i.e. $= 2$), then c is also even (i.e. $c = 2$), but

$$2^n + 2^n = 2^{n+1} = 2^2$$

is impossible.

If n is odd, recall that

$$a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + \cdots + (-1)^{n-1}b^{n-1}),$$

we also note that these two factors are different. Suppose it were true that $a + b = (a^{n-1} - a^{n-2}b + \cdots + (-1)^{n-1}b^{n-1})$, then $a^n + b^n = (a + b)^2$. However, $a^n + b^n \geq a^3 + b^3 > (a + b)^2$ for $a, b \geq 2$ and a, b not both 2. So $c^2 = a^n + b^n$ implies c^2 can be factorized into two different numbers. So $a + b = 1$ or $a + b = c^2$. $a + b = 1$ is impossible and $a + b = c^2 \implies c^2 = a^n + b^n > a + b = c^2$, it's also impossible.

So n must even, let $n = 2m$, then

$$(a^m)^2 + (b^m)^2 = c^2.$$

By the theorem on Pythagorean triple, there is $u, v \in \mathbb{N}$, $u > v$ such that

$$(a^m, b^m, c) = (u^2 - v^2, 2uv, u^2 + v^2).$$

Now $b^m = 2uv$, $2|b^m \implies 2|b$ and b is prime implies $b = 2$, so $2^{m-1} = uv$, so $u = 2^a$ and $v = 2^b$, $a + b = m - 1$ with $a > b \geq 0$. $a^m = (u + v)(u - v) = (2^a + 2^b)(2^a - 2^b)$, so a is even and hence $a = 2$. But $a = b = 2$ is also impossible in this case!

2.11 Metric Spaces

272. To be added.

273. Let $E_n = \{x : f^n(x) = 0\}$, then $\bigcup_{n=1}^{\infty} E_n = \mathbb{R}$. Now $\mathbb{R}^\circ \neq \emptyset \implies \mathbb{R}$ is of second category $\implies \mathbb{R}$ is not of first category $\implies$ for positive positive integer n, E_n is not nowhere dense, then

$$(\overline{E_n})^\circ \neq \emptyset. \tag{*}$$

What's more, E_n is closed, let $x \in \overline{E_n}$, then there is a sequence $\{x_n \in E_n\}$ that tends to x, then $f^{(n)}(x) = f^{(n)}(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f^{(n)}(x_n) = 0$, thus $x \in E_n$. As a result, (*) tells us $E_n^\circ \neq 0$, take $x_0 \in E_n^\circ$, then there is an open ball in E_n. As a ball on $\mathbb{R}$ is an open interval, we have shown that there is a open interval (a, b) such that

$$f^{(n)}(x) = 0,$$

integrating it n times will give us a polynomial of $n - 1$ degree.

274. **Method 1.** The first part is easy. For the second part, let $L = \overline{L}$, prove the claim that $L = \bigcap_{n=1}^{\infty} \left(\bigcup_{x \in L} B\left(x, \frac{1}{n}\right) \right)$.

Method 2. Let $f(x) = d(x, A)$, then f is continuous, now $x \in \overline{A} \iff f(x) = 0 \iff x \in f^{-1}(0)$. But $\{0\} = \cap_{n=1}^{\infty} [0, \frac{1}{n})$ and $[0, \frac{1}{n})$ is open in $[0, \infty)$, for all $n \in \mathbb{N}$. So

$$A = \overline{A} = f^{-1}\left(\bigcap_{n=1}^{\infty} \left[0, \frac{1}{n}\right) \right) = \bigcap_{n=1}^{\infty} f^{-1}\left[0, \frac{1}{n}\right).$$

275. Direct use of Contraction mapping theorem.

276. Suppose on the contrary, for all $\forall r > 0, \exists x \in M, \forall U \in \mathcal{U} = \{U_\alpha : \alpha \in A\}$, $B(x,r) \nsubseteq U_i$.

Let $r = \frac{1}{n}$, there is $x_n \in M$, $B(x_n, \frac{1}{n}) \nsubseteq U_\alpha$, for all $\alpha \in A$. As M is compact, x_{n_k} converges to x, for all $\epsilon > 0$, there is K_1 such that

$$k > K_1 \implies d(x_{n_k}, x) < \frac{\epsilon}{2}.$$

But the compactness tells us $x \in M$, so $x \in U_\alpha, \exists \alpha$. $\exists r > 0$, $B(x,r) \subseteq U_\alpha$.

Let $y \in B(x_{n_k}, \frac{1}{n_k})$, $d(y,x) \leq d(y_1, x_{n_k}) + d(x_{n_k}, x) < \frac{1}{n_k} + \frac{\epsilon}{2}$. Choose k large so that $d(y,x) < \epsilon$, take $\epsilon = r$, $y \in B(x,r)$, thus

$$B\left(x_{n_k}, \frac{1}{n_k}\right) \subseteq B(x,r) \subseteq U_\alpha \in \mathcal{U},$$

a contradiction.

277. Suppose on the contrary for all $x \in X$, $d\big(x, f(x)\big) > 0$, by compactness and continuity, there is a $x_0 \in X$, such that $c = f\big(x_0, f(x_0)\big) = \inf\{d\big(x, f(x)\big) : x \in X\}$. As a result, for all $x \in X$, $d\big(x, f(x)\big) \geq c$. Take $x = x_n = f^{(n)}(x_0)$, we have

$$c = d\big(x_0, f(x_0)\big) > d\big(x_n, f(x_n)\big) \geq c,$$

a contradiction.

278. Write

$$\mathbb{R} \setminus S = \left(\bigcup_{k=1}^{\infty} W_k\right) \setminus \left(\bigcup_{n=1}^{\infty} (W_n)^\circ\right) = \bigcup_{k=1}^{\infty} \left(W_k \setminus \left(\bigcup_{n=1}^{\infty} (W_n)^\circ\right)\right),$$

since W_k is closed, $\bigcup_{n=1}^{\infty} (W_n)^\circ$ is open, we see that $W_k \setminus \left(\bigcup_{n=1}^{\infty} (W_n)^\circ\right)$ is closed, hence

$$\begin{aligned}\left(\overline{W_k \setminus \left(\bigcup_{n=1}^{\infty} (W_n)^\circ\right)}\right)^\circ &= \left(W_k \setminus \left(\bigcup_{n=1}^{\infty} (W_n)^\circ\right)\right)^\circ \\ &\subseteq \big(W_k \setminus W_k^\circ\big)^\circ \\ &= \emptyset,\end{aligned}$$

so $\mathbb{R} \setminus S$ is of the first category, and hence S is dense in $\mathbb{R}$.

279. To be added.

280. Since $\{(-\infty, c), (c, \infty) : c \in (a,b)\}$ is a topological subbasis for $\mathcal{T}_{\text{usual}} \cap [a,b]$, it suffices to check that $f^{-1}(-\infty, c)$ and $f^{-1}(c, \infty)$ are open for all $c \in (a,b)$. Let's fix $c \in (a,b)$. λ_0 always means some element in Λ.

Suppose $f(x) > c$, by density there is $\lambda_0 > c$ so that $f(x) > \lambda_0$. Note that $f(x) > \lambda_0 \implies x \in X - \mathcal{O}_{\lambda_0}$. Combined with our assumption, $f(x) > c \implies \exists \lambda_0 > c, x \in X - \mathcal{O}_{\lambda_0}$. Conversely, suppose $\exists \lambda_0 > c, x \notin \mathcal{O}_{\lambda_0}$, then for all $\lambda \leq \lambda_0$, $x \notin \mathcal{O}_\lambda$, meaning that $\{\lambda \in \Lambda : x \in \mathcal{O}_\lambda\} \subseteq (\lambda_0, \infty)$, so $f(x) \geq \lambda_0 > c$. Combining the above two directions, we conclude

$$f^{-1}(c, \infty) = \cup_{\lambda \in (c,b)} (X - \mathcal{O}_\lambda) = X - \cap_{\lambda \in (c,b)} \mathcal{O}_\lambda.$$

Now $f^{-1}(c, \infty)$ is open iff $\cap_{\lambda \in (c,b)} \mathcal{O}_\lambda$ is closed. But we always have $\overline{\cap_{\lambda \in (c,b)} \mathcal{O}_\lambda} \subseteq \cap_{\lambda \in (c,b)} \overline{\mathcal{O}_\lambda}$. And by normal ascending property, $\cap_{\lambda \in (c,b)} \overline{\mathcal{O}_\lambda} \subseteq \cap_{\lambda \in (c,b)} \mathcal{O}_\lambda$, we conclude that $\overline{\cap_{\lambda \in (c,b)} \mathcal{O}_\lambda} \subseteq \cap_{\lambda \in (c,b)} \mathcal{O}_\lambda$, proving that $f^{-1}(c, \infty)$ is open.

Similarly, if $f(x) < c$, then there is $\lambda_0 < c$, $f(x) < \lambda_0$. So there must be $\lambda < \lambda_0$, $x \in \mathcal{O}_\lambda \subseteq \mathcal{O}_{\lambda_0}$, for some $\lambda_0 < c$. Conversely, if $x \in \mathcal{O}_{\lambda_0}$, for some $\lambda_0 < c$, then $f(x) \leq \lambda_0 < c$, hence $f^{-1}(-\infty, c) = \cup_{\lambda \in (a,c)} \mathcal{O}_\lambda$, which is of course open.

2.12 Linear Algebra

281. Suppose there are n eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of T, then let $v_1, v_2, \dots, v_n$ be the corresponding eigenvectors. Since

$$x \in \operatorname{range} T|_{\operatorname{span}(v_1,\dots,v_n)} \iff \exists a_i \in \mathbb{F}, x = T\left(\sum_{i=1}^{n} a_i v_i\right) \iff \exists a_i \in \mathbb{F}, x = \sum_{i=1}^{n} a_i \lambda_i v_i$$
$$\iff \begin{cases} x \in \operatorname{span}(v_1,\dots,v_n), & \text{all } \lambda_i \neq 0, \\ x \in \operatorname{span}_{i\neq j} v_i, & \lambda_j = 0, \end{cases}$$

we conclude $\operatorname{range} T|_{\operatorname{span}(v_1,\dots,v_n)} = \operatorname{span}_i v_i$ or $\operatorname{span}_{i\neq j} v_i$, in either case we have

$$k = \dim \operatorname{range} T \geq \dim \operatorname{range} T|_{\operatorname{span}(v_1,\dots,v_n)} \geq n-1 \implies k+1 \geq n.$$

282. To be added.

283. To be added.

284. To be added.

285. If $T = cI$, for some $c \in \mathbb{F}$, then clearly $ST = TS$ for all $S \in \mathcal{L}(V)$.

Conversely, assume $ST = TS$ for all $S \in \mathcal{L}(V)$. If $T \equiv 0$, we are done since $T = 0I$. Let's further assume $T \not\equiv 0$.

Now it suffices to prove every $v \in V$ is an eigenvector of T (then by problem 283, $T = cI$ for some $c \in \mathbb{F}$). Let $v \in V$, suppose v and Tv are linearly independent (automatically implying that $v, Tv \neq 0$), we try to show it is absurd (and the proof is completed). We can define a linear map $S \in \mathcal{L}(\operatorname{span}(v, Tv))$ by

$$S\big(a(v) + b(Tv)\big) = a(\alpha v) + b(Tv), \quad a, b \in \mathbb{F}, \alpha \neq 0,$$

then since $TSv = STv$, we conclude $T(\alpha v) = Tv \implies (\alpha - 1)Tv = 0$, taking $\alpha = 2$, a contradiction arises. So for any $v \in V$, v and Tv are linearly dependent, we are done.

286. To be added.

287. If $\dim V = 1$, then it is trivial without the hypothesis. For $\dim V \geq 2$ we try to use problem 283. For convenience, let $n = \dim V$. Let $v \in V \setminus \{0\}$, assume (v, Tv) is linearly independent, we try to derive a contradiction (hence (v, Tv) is always linearly dependent and hence all nonzero $v \in V$ is an eigenvalue, we are done). We extend the list to $(v, Tv, v_1, v_2, \dots, v_{n-2})$, the basis of V. Now $U = \operatorname{span}(v, v_1, v_2, \dots, v_{n-2})$ is an $n-1$ dimensional subspace and hence invariant under T from our hypothesis. In particular, $v \in U$, so $Tv \in U$, i.e. U is at least n dimensional, a contradiction.

288. To be added.

289. Clearly $\operatorname{null} T \subseteq \operatorname{null} ST$. If $\operatorname{null} T = \operatorname{null} ST$, we are done. Otherwise let $(t_1, t_2, \dots, t_m)$ be a basis of $\operatorname{null} T$, extend it to

$$(t_1, t_2, \dots, t_m, u_1, \dots, u_n),$$

the basis of $\operatorname{null} ST$. Clearly $Tu_i \neq 0$, if not $\operatorname{null} T$ will be of dimension at least $m+1$, impossible. Now $S(Tu_i) = 0$ (iff $Tu_i \in \operatorname{null} S$), for all i, it is natural to ask whether $(Tu_1, \dots, Tu_n)$ is linearly independent.

Now

$$\textstyle\sum_{i=1}^{n} a_i T u_i = 0, \exists a_i \implies T\left(\sum_{i=1}^{n} a_i u_i\right) = 0, \exists a_i \implies \sum_{i=1}^{n} a_i u_i \in \operatorname{null} T, \exists a_i,$$

hence $\sum_{i=1}^{n} a_i u_i \in \operatorname{span}(t_1, t_2, \ldots, t_m)$, but $(t_1, t_2, \ldots, t_m, u_1, \ldots, u_n)$ is linearly independent, $a_i = 0$, for all i. So $\operatorname{span}(Tu_1, \ldots, Tu_n) \subseteq \operatorname{null} S \implies n \leq \dim \operatorname{null} S$, and hence

$$\dim \operatorname{null} ST = m + n = \dim \operatorname{null} T + n \leq \dim \operatorname{null} T + \dim \operatorname{null} S.$$

290. To be added.

291. To be added.

292. To be added.

293. To be added.

294. (a) Clearly the intersection is non-empty. Now $\operatorname{range} T = \operatorname{range} T^2$ and $\operatorname{null} T = \operatorname{null} T^2$ (why? both derived from the same reasoning). We let $v \in \operatorname{null} T \cap \operatorname{range} T$, then we see that $Tu = v, \exists u \in V \implies T^2 u = 0$, so $u \in \operatorname{null} T^2 = \operatorname{null} T$, i.e. $Tu = 0$, thus $v = 0$. So the intersection is trivial.

If $V = \operatorname{null} T$ or $V = \operatorname{range} T$, then we are done. Ignoring these two trivial cases, we suppose $\dim V = n$, let $(v_1, v_2, \ldots, v_k)$ be the basis of $\operatorname{null} T$, expand the list such that $(v_1, \ldots, v_k, u_1, \ldots, u_{n-k})$ is the basis of V. Then clearly $u_1, \ldots, u_{n-k}$ are not necessarily vectors in $\operatorname{range} T$, we claim that

$$(v_1, \ldots, v_k, Tu_1, \ldots, Tu_{n-k}) \text{ is the basis of } V.$$

We see that

$$\begin{aligned}
\sum a_i v_i + \sum b_i T u_i = 0 &\implies \sum b_i T^2 u_i = 0 \implies T^2 \left(\sum b_i u_i\right) = 0 \\
&\implies \sum b_i u_i \in \operatorname{null} T^2 = \operatorname{null} T \implies \sum b_i u_i = \sum c_i v_i \\
&\implies b_i = 0, \forall i \implies \sum a_i v_i = 0 \\
&\implies a_i = 0, \forall i.
\end{aligned}$$

Indeed $(v_1, \ldots, v_k, Tu_1, \ldots, Tu_{n-k})$ is a list of linearly independent vectors, so $V = \operatorname{null} T + \operatorname{range} T$, but their intersection is trivial, we conclude that $V = \operatorname{null} T \oplus \operatorname{range} T$.

(b) Take $k = \dim V$, then $\operatorname{range} T^{\dim V} = \operatorname{range} T^{\dim V + 1}$, repeat all the statements above, we are done.

295. Let $(v_1, v_2, \ldots, v_k)$ be the basis of $\operatorname{null} A$.

For any $u \in \operatorname{null} QAP$, we have $QAPu = 0 \implies APu = 0 \implies Pu \in \operatorname{null} A$, so

$$Pu = \sum_{i=1}^{k} a_i v_i \implies u = \sum_{i=1}^{k} a_i P^{-1} v_i,$$

this tells us the list of vectors $B = (P^{-1}v_1, P^{-1}v_2, \ldots, P^{-1}v_k)$ spans $\operatorname{null} QAP$. It remains to show that B is linearly independent. Suppose that $\sum a_i P^{-1} v_i = 0$, then

$$P^{-1}\left(\sum a_i v_i\right) = 0 \implies \sum a_i v_i = 0 \implies a_i = 0.$$

So $\dim \operatorname{null} QAP = \dim \operatorname{null} A$, noting that both domains of A and QAP is $\mathbb{R}^n$, by rank-nullity theorem, we are done.

296. Note that A is an $m \times n$ matrix implies $A^T A$ is an $n \times n$ matrix, so both the domain of A and $A^T A$ is $\mathbb{R}^n$, thus by rank-nullity theorem,

$$\operatorname{rank} A^T A = \operatorname{rank} A \iff \dim \operatorname{null} A^T A = \dim \operatorname{null} A.$$

To prove so, we either prove that their null spaces are the same or prove their null spaces are spanned by the same number of linearly independent vector(s). Clearly $\operatorname{null} A^T A \supseteq \operatorname{null} A$. To prove $\operatorname{null} A \supseteq \operatorname{null} A^T A$, let $v \in \operatorname{null} A^T A$, then $A^T A v = 0$, so

$$0 = \langle v, 0\rangle = \langle v, A^T A v\rangle = \langle Av, Av\rangle = \|Av\|^2,$$

hence $v \in \operatorname{null} A$, we conclude that $\operatorname{null} A = \operatorname{null} A^T A$.

297. Assume $(I+S)x = 0$, then it is easy to deduce that $x = 0$, meaning that $I+S$ is injective, and hence invertible.

298. The ($\Rightarrow$) direction is clear since when $\lambda \in \sigma(A)$, one has $Av = \lambda v$ for some $v \neq 0$, then $A^k v = \lambda^k v \to 0 \implies |\lambda| < 1$ (we can find an k so that $|\lambda^k|\|v\| < \|v\|$), and as $\sigma(A)$ is at most finite, we conclude $\rho(A) < 1$.

For the ($\Leftarrow$) direction, we upper triangulize A by some $P \in GL_n(\mathbb{C})$, i.e.,

$$U := PAP^{-1} = \begin{pmatrix} \lambda_1 & b_{12} & \cdots & b_{1n} \\ 0 & \lambda_2 & \cdots & b_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

then $U\mathbf{e}_1 = \lambda_1 \mathbf{e}_1$, and $U^j \mathbf{e}_1 = \lambda_1^j \mathbf{e}_1$ and hence $|\lambda_1| < 1 \implies U^k \mathbf{e}_1 \to 0$. We complete the proof by induction, assume there is $k \in \mathbb{N}$ so that

$$\lim_{j\to\infty} U^j(\mathbf{e}_1), \dots, \lim_{j\to\infty} U^j(\mathbf{e}_{k-1}) = 0.$$

Then since $U(\mathbf{e}_k) = \sum_{i=1}^{k-1} b_{ik}\mathbf{e}_i + \lambda_k \mathbf{e}_k$, we have $U^{j+1}(\mathbf{e}_k) = \sum_{i=1}^{k-1} b_{ik} U^j(\mathbf{e}_i) + \lambda_k U^j(\mathbf{e}_k)$. For a vector $v \in \mathbb{C}^n$ let's denote $[v]_\ell$ to be its ℓ^{th} coordinate w.r.t. the usual basis, then one has

$$[U^{j+1}(\mathbf{e}_k)]_\ell = \sum_{i=1}^{k-1} b_{ik}[U^j(\mathbf{e}_i)]_\ell + \lambda_k [U^j(\mathbf{e}_k)]_\ell \implies \big|[U^{j+1}(\mathbf{e}_k)]_\ell\big| \le \sum_{i=1}^{k-1} |b_{ik}|\big|[U^j(\mathbf{e}_i)]_\ell\big| + |\lambda_k|\big|[U^j(\mathbf{e}_k)]_\ell\big|,$$

so $\overline{\lim}_{j\to\infty} \big|[U^{j+1}(\mathbf{e}_k)]_\ell\big| \le \big|\lambda_k\big| \,\overline{\lim}_{j\to\infty} \big|[U^j(\mathbf{e}_k)]_\ell\big| \implies \overline{\lim}_{j\to\infty} \big|[U^j(\mathbf{e}_k)]_\ell\big| = 0 \implies \lim_{j\to\infty} [U^j(\mathbf{e}_k)]_\ell = 0$. As this is true for $\ell = 1, 2, \dots, n$, so $\lim_{j\to\infty} U^j(\mathbf{e}_k) = 0$.

We conclude by induction that $\lim_{j\to\infty} U^j(\mathbf{e}_k) = 0$ for $k = 1, 2, \dots, n$. Since each vector in $\mathbb{R}^n$ is spanned by $\{\mathbf{e}_i\}_{i=1}^n$, we conclude $U^j \to 0$. Since $A^j = P^{-1}U^j P$, we conclude $A^j \to 0$.

299. To be added.

300. To be added.

301. The equivalence of (ii), (iii) is obvious, what is left is (i) $\Leftrightarrow$ (ii). Let P be the orthogonal projector onto range A[4]. Assume x_0 solves the LSP, write $r := b - Ax_0$, then

$$\begin{aligned} \|r\|^2 &= \|Pr\|_2^2 + \|r - Pr\|_2^2 \\ &= \|Pr\|_2^2 + \|b - Ax_0 - Pr\|_2^2 \\ &\ge \|Pr\|_2^2 + \|b - Ax_0\|_2^2 \\ &= \|Pr\|_2^2 + \|r\|_2^2, \end{aligned}$$

[4] Not assuming A has full rank.

we conclude $Pr = 0$, i.e., $r = b - Ax_0 \in (\operatorname{range} A)^\perp$. Conversely, assume $b - Ax_0 \in (\operatorname{range} A)^\perp$, then for each $x \in \mathbb{C}^n$,

$$\|b - Ax_0\|_2 \le \sqrt{\|b - Ax_0\|_2^2 + \|Ax\|_2^2} = \|b - A(x_0 - x)\|_2,$$

this shows that x_0 does solve the LSP.

We prove the last assertion now. Suppose A has full rank, let x_1, x_2 solve the LSP. By considering the polynomial

$$P(\epsilon) := \|b - [\epsilon A x_1 + (1-\epsilon)Ax_2]\|_2^2 = \|(Ax_1 - Ax_2)\epsilon + Ax_2 - b\|_2^2$$

in[5] ϵ, it is not hard to show $P''(\epsilon) = 0$, hence $Ax_1 = Ax_2$, and thus $A^*Ax_1 = A^*Ax_2$. But A has full rank, $x_1 = x_2$. Conversely, assume A does not have full rank, then A^*A is not injective, thus there is nonzero $x \in \mathbb{C}^n$, $A^*Ax = 0$, so $Ax = 0$. Now $x + x_0$ also solves the LSP, the solution to LSP is not unique.

302. To be added.

303. To be added.

304. To be added.

305. To be added.

306. To be added.

307. To be added.

308. To be added.

309. To be added.

310. To be added.

311. To be added.

312. To be added.

313. To be added.

2.13 Algebra

314. Let A, B be two proper subgroups of a group G, then $A \setminus B$ is nonempty (otherwise $B = G$). Let $a \in A \setminus B$, take $g \in G$, if $g \in A$, then clearly $ga \in A$. If $g \in B$, then $ga \in B \implies a \in B$, a contradiction. Hence $ga \in A$, this means $Ga \subseteq A$. But Ga is just a permutation of elements in G, i.e. $Ga = G$, hence $G \subseteq A$, a contradiction.

315. (a) Answer is 25, see (b) to get the general idea.

(b) Let $L(n,k) = |\{\sigma \in S_n : |\sigma| = k\}|$,

$$L(n,2) = \binom{n}{2} + \frac{\binom{n}{2}\binom{n-2}{2}}{2!} + \frac{\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}}{3!} + \cdots + \frac{\binom{n}{2}\binom{n-2}{2}\cdots\binom{n-2k}{2}}{(k+1)!}.$$

We are left to determine such least possible k, we need $n - 2k \ge 2$, so $k + 1 \le \dfrac{n}{2}$, thus the range of k is

$$1 \le k+1 \le \left[\frac{n}{2}\right] \implies 0 \le k \le \left[\frac{n}{2}\right] - 1,$$

[5] One can expand by the formula $\|a+b\|_2^2 = \|a\|_2^2 + \|b\|_2^2 + 2\operatorname{Re}\langle a, b\rangle$.

so the number is precisely

$$L(n,2)=\sum_{k=0}^{[\frac{n}{2}]-1}\frac{1}{(k+1)!}\prod_{r=0}^{k}\binom{n-2r}{2}.$$

(c) In general for $2\le p\le n$, when we have chosen a permutation $(i_1\, i_2\, \cdots\, i_p)$, we get total distinct permutations by multiplying $(p-1)!$, namely,

$$i_1\to(p-1)\text{ choices}\to(p-2)\text{ choices}\to\cdots\to 1\text{ choice (back to }i_1),$$

we generalize $L(n,2)$ by

$$L(n,p)=\sum_{k=0}^{[\frac{n}{p}]-1}\frac{1}{(k+1)!}\prod_{r=0}^{k}\left(\binom{n-pr}{p}(p-1)!\right)=\sum_{k=1}^{[\frac{n}{p}]}\frac{\left((p-1)!\right)^k}{(k)!}\prod_{r=0}^{k-1}\binom{n-pr}{p}.$$

To "verify" the result, lets count $L(n,3)$ in another way, we count this by considering a_n defined by

$$a_n=\frac{1}{3}\times\underbrace{\binom{n}{2}}_{\substack{\text{choose 2 elements}\\ \text{to form permutation}}}\quad\underbrace{\binom{2}{1}}_{\substack{\text{choose 1 of the first two}\\ \text{chosen numbers}}}\quad\underbrace{\binom{n-2}{1}}_{\substack{\text{choose another 1 to form permutation}\\ \text{with the number in 2C1}}}.$$

The factor $\frac{1}{3}$ is left there because we observe that every length 3 permutation can be written as product of 2 transpositions in exactly 3 ways.

Then clearly the number of ways to form order 3 permutation by multiplying k disjoint length 3 cycles is given by

$$\frac{\prod_{r=0}^{k-1}a_{n-3r}}{(k)!}=\frac{2^k}{(k)!}\prod_{r=0}^{k-1}\binom{n-3r}{3},$$

exactly the same summand appears.

316. The map $a(H\cap K)\overset{\varphi}{\mapsto}(aH,aK)$ is a group isomorphism between $G/(H\cap K)$ and $(G/H)\times(G/K)$. Well-definedness and injectivity are easy to check. We need to argue a little bit on surjectivity.

Let $(aH,bK)\in(G/H)\times(G/K)$, since $HK=G=G^{-1}=KH$, we can write $a=k_1h_1$ and $b=h_2k_2$, $h_i\in H$ and $k_i\in K$, $i=1,2$,

$$(aH,bK)=(k_1h_1H,h_2k_2K)=(k_1H,h_2K)=(k_1h_2H,Kh_2)=(k_1h_2H,k_1h_2K)=\varphi(k_1h_2H\cap K).$$

Thus the map φ is surjective.

Remark. *From above let H,K be normal subgroups of a group G, if HK forms a group, then $HK=KH$. Conversely, if $HK=KH$, then HK forms a group. So $\boxed{HK \text{ is a group if and only if } HK=KH}$.*

317. To be added.

318. Let $N=A\cap B$. We can check N is a subgroup of A and B. Let $x,y\in N$, then there are $a_i\in A$, $b_i\in B$ such that $x=a_1=b_1$, $y=a_2=b_2$, then clearly $xy^{-1}=a_1a_2^{-1}=b_1b_2^{-1}$, so $xy^{-1}\in N$.

It is natural to decompose AB as follows.

$$AB=\left(\bigcup_{i=1}^{[A:N]}a_iN\right)B=\bigcup_{i=1}^{[A:N]}a_iNB=\bigcup_{i=1}^{[A:N]}a_iB,$$

here $[H:K]$ denotes the number of left co-sets in H/K, we also let $\{a_i\}_{i=1}^{[A:N]} = A'$.

Let $a_p, a_q \in A'$. Suppose $a_pB \cap a_qB \neq \emptyset$, we let $b_r, b_s \in B$, such that

$$a_pb_r = a_qb_s \implies a_q^{-1}a_p = b_sb_r^{-1},$$

this tells us $a_q^{-1}a_p \in N = A \cap B \implies a_pN = a_qN \implies a_p = a_q$, as the representatives of co-sets have been fixed. We have proved all a_iB's are disjoint.

Since a_iB's are disjoint and $|a_iB| = |B|$ (elements inside a_iB are still distinct), the number of element(s) is

$$[A : A \cap B]|B| = \frac{|A||B|}{|A \cap B|}.$$

319. To be added.

320. It is clear that for each $x \in S_a$, $xa^k \in S_a$, so $x\langle a\rangle$ is contained in S_a, as a result,

$$S_a \supseteq x\langle a\rangle \supseteq \{x\} \implies S_a \supseteq \bigcup_{x\in S_a} x\langle a\rangle \supseteq S_a,$$

so $S_a = \sqcup_{i=1}^N x_i\langle a\rangle \implies |S_a| = Nn$ (exactly the same idea in Lagrange's theorem).

321. To be added.

322. To be added.

323. To be added.

324. To be added.

325. To be added.

326. To be added.

327. Observe that $H \subseteq Z(G) \iff ghg^{-1} = h, \forall g \in G, \forall h \in H \iff |\operatorname{Orb}(h)| = 1, \forall h \in H$. Here $\operatorname{Orb}(h) = \{ghg^{-1} : g \in G\}$ and the action is conjugate action. Normality of H implies that $\operatorname{Orb}(h) \subseteq H$, so to prove $|\operatorname{Orb}(h)| = 1$, it suffices to prove that $\operatorname{Orb}(h) \neq H$.

Now $\operatorname{Orb}(h) = H$ implies there is $g \in G$ such that $ghg^{-1} = e \in H$, so $h = e$. That is to say, if $H \ni h \neq e$, then $\operatorname{Orb}(h) = 1$. So clearly $Z(G) \supseteq H \setminus \{e\}$. However it is clear that $e \in Z(G)$, so $Z(G) \supseteq H$.

Remark. *We can also consider the action of group H on G, but in this way we see the assumption that H is normal becomes unhelpful.*

328. To be added.

329. We show that if $\vec{a}, \vec{b}$ and $\vec{a}', \vec{b}'$ are pairs of linearly independent vectors in $\mathbb{R}^2$ such that they generate the same discrete subgroup in $\mathbb{R}^2$ in the sense that

$$\mathbb{Z}\vec{a} + \mathbb{Z}\vec{b} = \mathbb{Z}\vec{a}' + \mathbb{Z}\vec{b}',$$

then the transition matrix from $(\vec{a}, \vec{b})$ to $(\vec{a}', \vec{b}')$ has determinant ± 1.

For simplicity we identify $\vec{a}, \vec{b}, \vec{a}', \vec{b}'$ with the symbol $a, b, a', b' \in \mathbb{R}^2$ respectively. Since there are $A, B, C, D \in \mathbb{Z}$ such that

$$a' = Aa + Bb \quad \text{and} \quad b' = Ca + Db,$$

we are left to show that $P := \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ has determinant ± 1.

By assumption one has

$$\begin{aligned}\mathbb{Z}a + \mathbb{Z}b &= \mathbb{Z}a' + \mathbb{Z}b' \\ &= \mathbb{Z}(Aa + Bb) + \mathbb{Z}(Ca + Db) \\ &= \cup_{m,n\in\mathbb{Z}}\{(mA + nC)a + (mB + nD)b\}, \qquad (*)\end{aligned}$$

this is the same as saying that for any $u, v \in \mathbb{Z}$, we can find $m, n \in \mathbb{Z}$ such that

$$\begin{cases} u = mA + nC \\ v = mB + nD \end{cases} \iff \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} = P\begin{pmatrix} m \\ n \end{pmatrix},$$

this shows that $P : \mathbb{Z}^2 \to \mathbb{Z}^2$ is surjective.

We first assume A, B, C, D are nonzero, for the case some of them are zero, the same approach will do. We observe that

$$P\begin{pmatrix} D \\ -B \end{pmatrix} = D\begin{pmatrix} A \\ B \end{pmatrix} - B\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} DA - BC \\ 0 \end{pmatrix}.$$

and also that

$$Pu \in \mathbb{Z} \times \{0\} \iff u \in \mathbb{Z}\begin{pmatrix} D \\ -B \end{pmatrix},$$

this is because for $u = \binom{x}{y} \in \mathbb{Z}^2$, $Pu \in \mathbb{Z} \times \{0\} \iff xB + yD = 0 \iff y = -xB/D \iff x = Dn, y = -nB$ for some $n \in \mathbb{Z}$, the last one follows from the observation that $\gcd(B, D) = 1$ (by (*), otherwise P cannot be surjective). Now we take an $x_0 \in \mathbb{Z}^2$ such that $Px_0 = \binom{1}{0}$, then there is an integer k such that $x_0 = k\binom{D}{-B}$, thus

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = Px_0 = kP\begin{pmatrix} D \\ -B \end{pmatrix} = k\begin{pmatrix} DA - BC \\ 0 \end{pmatrix},$$

and hence $1 = k(DA - BC) \implies |DA - BC| = 1$.

330. (a) Let Γ be a subgroup of $\mathbb{R}$ under addition. If Γ is dense in $\mathbb{R}$, we are done. Otherwise, there is a point $x_0 \in \mathbb{R}$ that is neither a point nor a limit point of Γ, i.e. there is $\epsilon > 0$ such that $B(x_0, \epsilon) \cap \Gamma = \emptyset$. If $\Gamma = \{0\}$, then it is clearly discrete. Let's assume Γ is not trivial. If there are distinct $a, b \in \Gamma$ such that $|a - b| < \epsilon/2$, then there will be a $k \in \mathbb{Z}$ such that $a + k(a - b) \in B(x_0, \epsilon)$, a contradiction. We conclude $|a - b| \geq \epsilon/2$, for all distinct $a, b \in \Gamma$, hence Γ is discrete.

(b) If $\mathbb{Z} + \mathbb{Z}\sqrt{2}$ is not dense in $\mathbb{Z}$, then it must a discrete subgroup of $(\mathbb{R}, +)$, and any additive discrete subgroup of $\mathbb{R}$ must be of the form $\mathbb{Z}a$, for some $a > 0$, thus there is $a > 0$ such that $\mathbb{Z} + \mathbb{Z}\sqrt{2} = \mathbb{Z}a$. Clearly there are nonzero integers m, n such that

$$1 + 0\sqrt{2} = ma \quad \text{and} \quad 0 + 1\sqrt{2} = na \implies a \in \mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}),$$

a contradiction.

(c) It suffices to construct $H' = \{\theta \in \mathbb{R} : \rho_\theta \in H\}$ and $G' = \{\theta \in \mathbb{R} : \rho_\theta \in G\}$.

331. (a) Let G be a finite subgroup of $F^\times$. Observe that $u := \prod_{g\in G} g$ has order $L := \operatorname{lcm}\{|g| : g \in G\}$. That is, u is an element in G which has largest order. Now $\langle u\rangle := \{u^k : k \in \mathbb{Z}\} \subseteq \{x \in G : x^L = 1\}$. Since F is a field, there are at most L solutions in F for the equation $x^L = 1$, hence $\langle u\rangle = \{x \in G : x^L = 1\}$. For each $g \in G$, $|g|\,|\,L$, so $g^L = 1$, thus $g \in \langle u\rangle$. We conclude $G = \langle u\rangle$.

(b) It suffices to show the statement for cyclic subgroups of $F^\times$. The statement is trivial for $n = 1$. We prove by induction on n. Let there be at most one cyclic subgroup of order $1, 2, \ldots, n - 1$. Assume $|\langle x\rangle| = |\langle y\rangle| = n$, then $|x| = |y|$ and for each $k \geq 2$, we have $\langle x^k\rangle = \langle y^k\rangle$. So $x = x^3(x^2)^{-1} \in \langle x^3\rangle\langle x^2\rangle = \langle y^3\rangle\langle y^2\rangle \subseteq \langle y\rangle \implies \langle x\rangle \subseteq \langle y\rangle$. Interchanging x and y, we have $\langle x\rangle = \langle y\rangle$, as desired.

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