Midterm Review: More Problems on Limit Superior and Inferior

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Example 1, 2 and 3 demonstrate the same idea and same technique with increasing level of complexity, these 3 examples may not provide you sufficient technique to tackle the most challenging problems in the midterm exam of this course. Thus Example 4 to Example 7 aim at providing you the examples with various possible technique that may arise in problems related to lim sup and lim inf.

Specifically, in Example 4 and 5 we will try to refine the condition like

$$\overline{\lim} < ???$$

for which the problem is not readily solvable without any adjustment on the bound "???". Example 6 and 7 are miscellaneous.

Example 1. Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive numbers such that $\sum_{n=1}^{\infty} b_n = \infty$, show that

$$\overline{\lim}\, \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \le \overline{\lim}\, \frac{a_n}{b_n}. \tag{1}$$

Solution. If $\overline{\lim}\, a_n/b_n = \infty$, then inequality (1) holds obviously.

Suppose now $\overline{\lim}\, a_n/b_n < \infty$, we fix an $\alpha \in \mathbb{R}$ such that $\overline{\lim}\, a_n/b_n < \alpha$, then there is an N such that

$$n > N \implies \frac{a_n}{b_n} < \alpha.$$

Thus we have $a_n < \alpha b_n$ and

$$\begin{aligned}\frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} &= \frac{a_1 + \cdots + a_N + a_{N+1} + \cdots + a_n}{b_1 + \cdots + b_n} \\ &< \frac{a_1 + \cdots + a_N}{b_1 + \cdots + b_n} + \frac{\alpha(b_{N+1} + \cdots + b_n)}{b_1 + \cdots + b_n}.\end{aligned}$$

By taking $\overline{\lim}$ on the ends of the above inequality we have

$$\overline{\lim}\, \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \le 0 + \alpha \cdot 1 = \alpha.$$

Since $\alpha > \overline{\lim}\, a_n/b_n$ is arbitrary, we taking $\alpha \to (\overline{\lim}\, a_n/b_n)^+$ to get

$$\overline{\lim}\, \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \le \overline{\lim}\, \frac{a_n}{b_n}. \qquad (\hat{-}_{\triangledown}\hat{-})^{♪\sim}$$

> **Example 2 (2003 Midterm (L1)).** Let $\{a_n\}$ be a sequence of real numbers. show that
> $$\limsup_{n\to\infty} \frac{a_n}{n^2} \le \limsup_{n\to\infty} \frac{a_{n+1} - a_n}{2n+1}.$$

Solution. If $\overline{\lim}\, \dfrac{a_{n+1} - a_n}{2n+1} = \infty$, then we are done.

Suppose now $\overline{\lim}\, \dfrac{a_{n+1} - a_n}{2n+1} < \infty$, then we fix an $\alpha \in \mathbb{R}$ such that

$$\overline{\lim}\, \frac{a_{n+1} - a_n}{2n+1} < \alpha,$$

then there is an N such that

$$k \ge N \implies \frac{a_{k+1} - a_k}{2k+1} < \alpha \implies a_{k+1} - a_k < \alpha(2k+1).$$

Taking $\sum_{k=N}^{n-1}$ on both sides we have for every $n > N$,

$$\sum_{k=N}^{n-1} (a_{k+1} - a_k) < \sum_{k=N}^{n-1} \alpha(2k+1),$$

which is the same as

$$a_n - a_N < \alpha\left((n + N - 1)(n - N) + n - N\right).$$

Diving both sides by n^2 we have

$$\frac{a_n}{n^2} - \frac{a_N}{n^2} < \alpha\left(\frac{(n + N - 1)(n - N)}{n^2} + \frac{n - N}{n^2}\right).$$

Finally by taking $\overline{\lim}$ on both sides, we obtain

$$\overline{\lim}\, \frac{a_n}{n^2} \le \alpha,$$

but this is true for every (fixed) $\alpha > \overline{\lim}\, \frac{a_{n+1} - a_n}{2n+1}$, so by taking

$$\alpha \to \left(\overline{\lim}\, \frac{a_{n+1} - a_n}{2n+1}\right)^+$$

we have

$$\overline{\lim}\, \frac{a_n}{n^2} \le \overline{\lim}\, \frac{a_{n+1} - a_n}{2n+1}. \qquad (\hat{-}_{\triangledown}\hat{-})^{♪\sim}$$

Example 3 (2011 Midterm).

(a) Let $\{x_n\}$ be a sequence of real numbers. Prove that

$$\limsup_{n\to\infty} \frac{nx_{n+1} + (n+1)x_n}{n+1} \le \limsup_{n\to\infty} \big((n+1)x_{n+2} + 2x_{n+1} - (n+1)x_n\big).$$

(b) Give an example of a sequence $\{x_n\}$ of reals s.t. $\lim\limits_{n\to\infty} \frac{nx_{n+1} + (n+1)x_n}{n+1}$ exists but $\lim\limits_{n\to\infty} \big((n+1)x_{n+2} + 2x_{n+1} - (n+1)x_n\big)$ does not exist.

Solution. (a) If $\limsup\limits_{n\to\infty} \big((n+1)x_{n+2} + 2x_{n+1} - (n+1)x_n\big) = \infty$, then we are done.

Suppose now $\limsup\limits_{n\to\infty} \big((n+1)x_{n+2} + 2x_{n+1} - (n+1)x_n\big) < \infty$, then we fix an $\alpha \in \mathbb{R}$ such that

$$\limsup_{n\to\infty} \big((n+1)x_{n+2} + 2x_{n+1} - (n+1)x_n\big) < \alpha,$$

so there is an N such that

$$k \ge N \implies (k+1)x_{k+2} + 2x_{k+1} - (k+1)x_k < \alpha.$$

How is the term on the LHS related to $A_k := kx_{k+1} + (k+1)x_k$? We note that LHS is nothing but $A_{k+1} - A_k$, thus we have

$$A_{k+1} - A_k < \alpha,$$

therefore we have $A_n - A_N = \sum_{k=N}^{n-1}(A_{k+1} - A_k) < (n-N)\alpha$, and hence

$$\frac{A_n}{n} - \frac{A_N}{n} < \frac{n-N}{n}\alpha,$$

by taking $\overline{\lim}$ on both sides, we have

$$\overline{\lim}\, \frac{A_n}{n} \le \alpha.$$

Now the rest is routine!

(b) We choose $x_n = (-1)^n$, then

$$\frac{nx_{n+1} + (n+1)x_n}{n+1} = \frac{-n(-1)^n + (n+1)(-1)^n}{n+1} = \frac{(-1)^n}{n+1}$$

and

$$(n+1)x_{n+2} + 2x_{n+1} - (n+1)x_n = (-1)^n(n+1-2-(n+1)) = -2(-1)^n.$$

The former one converges to 0 but the later one diverges. (^▽^)♪~

Example 4 (2004 Midterm (L2)). Let $a_k \geq 0$ for $k = 1, 2, 3, \dots$ and

$$\limsup_{k\to\infty} a_k^{1/\ln k} < \frac{1}{e},$$

prove that $\displaystyle\sum_{k=1}^{\infty} a_k$ converges.

Solution. *First attempt*: Since $\overline{\lim}\, a_k^{1/\ln k} < 1/e$, there is a K such that

$$k > K \implies a_k^{1/\ln k} < 1/e.$$

And thus $k > K \implies a_k < 1/k$, which provides no information to the convergence of $\sum_{k=1}^{\infty} a_k$. We notice that the bound $1/e$ can be *shrink slightly*.

Second attempt: By continuity, there is an $\epsilon > 0$ such that $\overline{\lim}\, a_k^{1/\ln k} < \frac{1}{e^{1+\epsilon}} < \frac{1}{e}$, hence there is a K,

$$k > K \implies a_k^{1/\ln k} < \frac{1}{e^{1+\epsilon}},$$

and hence $k > K \implies a_k < 1/k^{1+\epsilon}$, and thus $\sum_{k=1}^{\infty} a_k$ converges. (^▽^)♪~

Example 5. Let $\{a_n\}$ be strictly increasing and unbounded, we define

$$s = \overline{\lim_{n\to\infty}} \frac{\ln n}{\ln a_n}.$$

Let $t > 0$, show that the series $\sum_{n=1}^{\infty} a_n^{-t}$ converges for $t > s$ and diverges for $t < s$.

Solution. Let $t > 0$. If $s = \infty$, then none of $t \in \mathbb{R}$ can satisfy $t > s$, it follows that the statement has nothing to prove.

Suppose now that $s < \infty$ and $t > s = \overline{\lim}_{n\to\infty} \ln n / \ln a_n$.

First attempt: By definition there is an N such that

$$n > N \implies t > \frac{\ln n}{\ln a_n}.$$

It follows that $t \ln a_n > \ln n$, and thus $a_n^{-t} < 1/n$. But then no conclusion can be made since $\sum 1/n = \infty$.

Second attempt: We need to refine the condition that $t > s$, we find that by continuity there is an $\epsilon > 0$ such that $\frac{t}{1+\epsilon} > s$, hence there is an N,

$$n > N \implies \frac{t}{1+\epsilon} > \frac{\ln n}{\ln a_n}.$$

On simplification, the last inequality becomes $a_n^{-t} < 1/n^{1+\epsilon}$, so $\sum a_n^{-t}$ converges by Comparison Test.

Suppose now $t < s$, then since s is a subsequential limit, we can find $\{n_k\}$ such that for every k, $t < \ln n_k / \ln a_{n_k}$. Therefore

$$a_{n_k}^{-t} > 1/n_k \iff n_k a_{n_k}^{-t} > 1.$$

Since $\{a_n^{-t}\}$ is decreasing sequence of positive numbers, by the fact that

$$\{b_n > 0\} \text{ decreasing and } \sum b_n < \infty \implies \lim_{n\to\infty} n b_n = 0$$

from Example 1 of tutorial note 4.5. We conclude $\sum a_n^{-t}$ diverges. (^▽^)♪~

Example 6 (2006 Midterm). Let $\{x_n\}$ be a bounded sequence of real numbers such that

$$\lim_{n\to\infty}(x_{n+1} - x_n) = 0, \quad \liminf_{n\to\infty} x_n = a \quad \text{and} \quad \limsup_{n\to\infty} x_n = b.$$

Show for every $c \in [a, b]$, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $\lim_{i\to\infty} x_{n_i} = c$.

Solution. Let $c \in [a, b]$. We further assume $c \neq a, b$ as otherwise we are done. Now we need to show that in every **small**[1] neighborhood $(c - r, c + r)$ of c, we can always find an element in $\{x_1, x_2, \ldots\}$.

To do this, let's for the sake of contradiction suppose there is a small neighborhood $(c - r, c + r)$ of c which contains **none** of x_n's. Then by the condition $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$, there is an $N \in \mathbb{N}$,

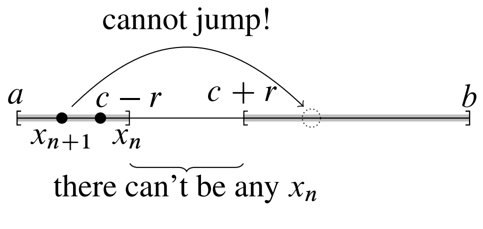
$$n > N \implies |x_{n+1} - x_n| < r.$$

Now the sequence $\{x_n\}_{n>N}$ cannot jump too far away from x_n to x_{n+1}.

Case 1 ($x_{N+1} \in [a, c - r]$). In this case,

$$\begin{aligned} x_{N+1} \in [a, c-r] &\implies x_{N+2} \in [a, c-r] \\ &\implies x_{N+3} \in [a, c-r] \\ &\implies \cdots \\ &\implies x_n \in [a, c-r] \end{aligned}$$

for every $n \geq N + 1$, so $\{x_n\}_{n \geq N+1}$ is trapped in $[a, c - r]$, then $\overline{\lim}\, x_n \leq c - r < b$, a contradiction.



[1] Let's define, by small, to mean $(c - r, c + r) \subseteq [a, b]$ such that $a < c - r$ and $c + r < b$.

Case 2 ($x_{N+1} \in [c+r, b]$). In this case the argument in case 1 carries over, and we arrive to the contradiction that $\underline{\lim}\, x_n \geq c + r > a$.

In conclusion, every small neighborhood of c contains one of x_n's. Let $K \in \mathbb{N}$ be such that

$$i > K \implies \left(c - \frac{1}{i}, c + \frac{1}{i}\right) \subseteq [a, b],$$

then there is an $x_{n_i} \in (c - \frac{1}{i}, c + \frac{1}{i})$. Now $\lim_{i\to\infty} x_{n_i} = c$. (^▽^)♪~

Example 7 (2008 Midterm).

(a) Give an example of a sequence $\{a_n\}$ of real numbers such that

$$\lim_{n\to\infty}(a_n - 2a_{n+1} + a_{n+2}) = 0 \quad \text{and} \quad \lim_{n\to\infty}(a_n - a_{n+1}) \neq 0.$$

(b) Let $\{a_n\}$ be a <u>*bounded*</u> sequence of real numbers and

$$\lim_{n\to\infty}(a_n - 2a_{n+1} + a_{n+2}) = 0.$$

(i) Prove that $\limsup_{n\to\infty}(a_n - a_{n+1}) \leq 0$.

(ii) Using (i) or otherwise, prove that $\liminf_{n\to\infty}(a_n - a_{n+1}) \geq 0$. Determine $\lim_{n\to\infty}(a_n - a_{n+1})$.

Solution. (a) The example $a_n = n$ will do.

(b) (i) Suppose that $\overline{\lim}(a_n - a_{n+1}) > 0$, then there is an $\epsilon_0 > 0$ such that

$$\overline{\lim}(a_n - a_{n+1}) > \epsilon_0.$$

Since $\overline{\lim}(a_n - a_{n+1})$ is a subsequential limit, there is $\{n_k\}$ such that

$$a_{n_k} - a_{n_k+1} > \epsilon_0 \tag{2}$$

for every k.

On the other hand, by the hypothesis for every $\epsilon > 0$ there is an $N = N(\epsilon)$ such that

$$n > N \implies a_n - a_{n+1} < a_{n+1} - a_{n+2} + \epsilon. \tag{3}$$

We will apply the estimate (3) successively to (2). To this end, we choose K large so that $k > K = K(\epsilon) \implies n_k > N$, then we have,

$$k > K, j \in \mathbb{N} \quad \implies \quad \begin{aligned} \epsilon_0 &< a_{n_k} - a_{n_k+1} \\ &< a_{n_k+1} - a_{n_k+2} + \epsilon \\ &< \cdots \\ &< a_{n_k+j} - a_{n_k+j+1} + j\epsilon. \end{aligned}$$

Therefore ignoring the index n_k we get a telescoping sequence in j.

For a fixed $p \in \mathbb{N}$, it is temping to do $\sum_{j=1}^{p}$ on both sides of the last inequality. By doing that we get for every $k > K$,

$$p\left(\epsilon_0 - \frac{p+1}{2}\epsilon\right) = p\epsilon_0 - \frac{p(p+1)}{2}\epsilon < a_{n_k+1} - a_{n_k+p+1}.$$

We can choose $\frac{p+1}{2}\epsilon = \frac{\epsilon_0}{2}$ at the beginning, then for $k > K(\epsilon)$ we get

$$\frac{p\epsilon_0}{2} < a_{n_k+1} - a_{n_k+p+1}.$$

Note that the choice of ϵ depends on p, thus K depends on p, we conclude for every p, there is an index k_p such that

$$\frac{p\epsilon_0}{2} < a_{k_p+1} - a_{k_p+p+1},$$

this contradicts the boundedness of $\{a_n\}$.

(ii) We can replace the sequence in (b) (i) by $-a_n$ to get

$$-\underline{\lim}(a_n - a_{n+1}) = \overline{\lim}((-a_n) - (-a_{n+1})) \leq 0,$$

and this becomes $\underline{\lim}(a_n - a_{n+1}) \geq 0$. Therefore we have $0 \leq \underline{\lim}(a_n - a_{n+1}) \leq \overline{\lim}(a_n - a_{n+1}) \leq 0$, and thus

$$\lim_{n\to\infty}(a_n - a_{n+1}) = 0. \qquad (\hat{-}_{\triangledown}\hat{-})^{♪\sim}$$